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Nonlinear CUB models

Marica Manisera, Paola Zuccolotto

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Marica Manisera and Paola Zuccolotto*

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Summary: A general statistical model for ordinal or rating data, which includes some existing approaches as special cases, is proposed. The focus is on the CUB models and a new class of models, called Nonlinear CUB, which generalize CUB. In the framework of the Nonlinear CUB models, it is possible to express a transition probability, i.e. the probability of increasing one rating point at a given step of the decision process. Transition probabilities and the related transition plots are able to describe the state of mind of the respondents about the response scale used to express judgments. Unlike classical CUB, the Nonlinear CUB models are able to model decision processes with non-constant transition probabilities.

Keywords: CUB models; decision process; latent variables; ordinal data; rating data; transition probability; Eurobarometer

1 Introduction

The latent variable approach is very wide, covers a variety of methods and techniques and can be applied in several fields (see, for example, Borsboom et al., 2003 and the references therein). In this paper, we propose a general statistical model for ordinal data that provides insights into the cognitive mechanism connected to the latent variables driving individuals' responses on a rating scale. In our model, the variety of complex interactions determining the final response is summarized by two components that, borrowing the terminology of the CUB models (Piccolo, 2003; D'Elia and Piccolo, 2005), can be called *feeling* and *uncertainty*. The feeling component accounts for reasoning and logical thinking as well as the set of emotions, perceptions, and subjective evaluations that individuals have with regard to the latent trait being evaluated. In our proposal, this is the outcome of a step-by-step decision process. The uncertainty component is concerned with the indecision inherently present in any human choice and does not generally depend on the individuals' position on the latent variable. A very interesting feature of the general model proposed in this paper is given by the so-called *transition probability*, defined as the probability of increasing one rating point at a given step of the decision process. Transition probabilities allow us to understand and describe the individuals' mental stance towards the response scale. The proposed general statistical model includes several approaches existing in the literature as special cases. In particular, we focus on the CUB models that were recently introduced to analyse ordinal data (starting with Piccolo, 2003, who introduced a mixture of two latent components to model ranking or rating data, and D'Elia, 2000, who previously suggested motivating a binomial component for exploring feeling in the paired comparisons model when the preferences are expressed by ranks) and generalize them by introducing a new class of models, called Nonlinear CUB, which, unlike classical CUB, are able to describe decision processes with non-constant transition probabilities.

2 Basic idea: the unconscious decision process in CUB and Nonlinear CUB models

In this paper, we propose a framework for the Decision Process (DP) which leads an individual to express a rating r about a latent trait on a given ordinal scale. In the followings, we refer to ratings as Likert-scaled responses, using the concept of Likert-type item in its broadest sense of ordered categorical item. In our proposal, r is generated by a latent DP composed of a feeling and an uncertainty component,

*University of Brescia, c.da S. Chiara 50 - 25122 Brescia, Italy.
email: marica.manisera@eco.unibs.it, paola.zuccolotto@eco.unibs.it

Table 1: DP of CUB models - Feeling approach (example with $m = 5$)

Elementary judgments	Positive sensation? Yes or no?				
Number of elementary judgments	$T = m - 1 = 4$				
Number of 'Yes' responses	0	1	2	3	4
Corresponding rating r_T	1	2	3	4	5

consistently with the founding idea of the CUB models. The idea is that the feeling component should take into account any reasoned assessment, as well as the set of emotions, sentiments and perceptions logically connected with the object being evaluated, while the uncertainty component accounts for other elements, such as, for example, the unconscious willingness to delight the interviewer or the indecision deriving from the difficulty one can find in evaluating some specific objects about which he/she has not a definite opinion. The two components result from two different approaches that individuals can follow to finally express their judgment. They are not necessarily consecutive but possibly coexist in the DP.

The *feeling approach* determines the feeling component of the DP and consists of a step-by-step process, called *feeling path*. The feeling path proceeds through T steps; at each step t an elementary judgement is given, which leads to a provisional rating r_t, so that the last rating r_T results from the accumulation of T elementary judgments. The uncertainty component of the DP is instead related to the *uncertainty approach*, which, for several reasons, leads an individual to formulate a completely random rating. The rating expressed by the respondent to the interviewer is derived by the feeling or the uncertainty approach, with given probabilities. However, for each answer, we do not know whether the respondent has decided according to the former or the latter approach.

As shown in the following two examples, this DP is suitable for explaining the generation of ratings distributed as postulated both by the CUB models (Piccolo 2003), and by the Nonlinear CUB models, which are proposed in this paper.

Example 1 - DP of CUB models: Suppose a person is asked to express a judgment about his/her satisfaction with a product by using a Likert scale from 1 to $m = 5$. In the feeling approach, the respondent asks him/herself for $T = m-1 = 4$ times 'Do I have a positive sensation about this product? Yes or no?' and gives a quick and instinctive response each time. The idea underlying this mechanism is that, while reasoning, positive and negative sensations come to mind randomly and disorderly, according to the individual's experience about the latent trait being evaluated. In the end, 1 plus the total number of 'Yes' responses is the last rating r_T of the feeling path (Table 1). In the uncertainty approach, for a wide variety of reasons (for example the respondent may not be completely confident on his reasoning, or he/she could be reluctant to let the interviewer know his real opinion), the rating is drawn from a discrete Uniform distribution in $(1, 2, \cdots, 5)$. The expressed rating can be formulated by the feeling or the uncertainty approach with probabilities π and $1 - \pi$, respectively. The distribution of the ratings assumed by the CUB models (Piccolo, 2003) is consistent with this unconscious mechanism.

Example 2 - DP of Nonlinear CUB models: As in Example 1, suppose a person is asked to express a judgment about his/her satisfaction with a product by using a Likert scale from 1 to $m = 5$. In the feeling approach, the person unconsciously asks him/herself for $T > m-1$ times 'Do I have a positive sensation about this product? Yes or no?' and gives a quick and instinctive response each time. For example, let T be equal to 8. In the end, the last rating of the feeling path is still based on the total number of 'Yes' responses, as in Example 1, but in an 'asymmetric' way. So, for example, zero 'Yes' responses can lead to the rating $r_T = 1$; one or two 'Yes' responses to the rating $r_T = 2$; three, four, five or six 'Yes' responses to the rating $r_T = 3$; seven 'Yes' responses to the rating $r_T = 4$ and, finally, eight 'Yes' responses to the highest rating $r_T = 5$ (Table 2). In other words, according to this DP, the respondent does not consider all the ratings on the Likert scale in the same way, and, in this example, moving from rating 3 to rating 4 is more difficult than moving from rating 1 to rating 2. As in Example 1, the final response can be r_T or a random rating resulting from the uncertainty approach, with probabilities π and $1-\pi$, respectively. Therefore, the Nonlinear CUB models differ from CUB only in the feeling approach. This basic idea is the foundation of the Nonlinear CUB models that constitute the proposal of this paper and will be more rigorously formalized in Section 4.

Table 2: DP of NLCUB models - Feeling approach (example with $m = 5$ and $T = 8$)

Elementary judgments	Positive sensation? Yes or no?								
Number of elementary judgments	$T = 8$ $(> m - 1)$								
Number of 'Yes' responses	0	1	2	3	4	5	6	7	8
Corresponding rating r_T	1	2		3				4	5

3 A general framework for the decision process

In this section, we describe a possible general framework for modelling the cognitive process driving individuals' responses to survey questions (items) asking for ratings with ordered response levels. Several authors support the intersection of cognitive psychology and survey methodology, which is useful for determining how the respondents' mental information processing influences their response process to survey questions and then the quality of collected data (see, for example, Belli et al., 2007; Tourangeau et al., 2000; Willis, 2008; van der Maas et al., 2011; Embretson and Gorin, 2001; Hines, 1993). In this paper, we do not want to propose a formal model of human cognition; we just describe a possible framework for the DP that lies between item administration and item response. In our view, such a framework is important to justify, understand and interpret the statistical model proposed in Section 4. In addition, it is very general, and other existing statistical models belong to this framework. We will formalize the DP of the responses to a single item, thus neglecting possible relationships among items. However, a multivariate generalization can be easily obtained.

We denote with r the Likert-scaled rating assigned to an item asking for a judgment on a latent trait. We assume that r can equal the last rating r_T formulated at the end of the feeling path or the random rating q generated following the uncertainty approach, with given probabilities. Formally, r_T is the last term of a step-by-step sequence of T intermediate provisional ratings $r_1, \cdots, r_T$, which are realizations of a sequence of random variables $R_1, \cdots, R_T$. The generic R_t, $t = 1, \ldots, T$, the random variable generating the rating at the t-th step of the DP, is given by

$$R_t = d(W_t) = d\left(f\left(X_1, \cdots, X_t\right)\right),$$

where $d(\cdot)$ is a 'Likertization' function, mapping the domain of W_t into the space $(1, \cdots, m)$, $f(\cdot)$ is an accumulating function, mapping the Cartesian product of the domains of $X_1, \cdots, X_t$ into $\mathbb{R}$, and $X_1, \cdots, X_t$, is an *iid* sequence of random variables describing t elementary judgments formulated throughout the feeling path up to step t (e.g., the positive or negative sensations of Examples 1 and 2). According to the uncertainty approach of the DP, which is related to the uncertainty component, the response is given by a random rating q drawn from a random variable Q with domain $(1, \cdots, m)$, independent on R_T. The final observed rating r is the realization of a random variable R defined as a mixture of R_T and Q with weight π, so that

$$r = \begin{cases} r_T & \text{with probability } \pi \\ q & \text{with probability } (1 - \pi) \end{cases}.$$

The following scheme summarizes the functioning of the proposed model.

A) Feeling Approach

1. *Elementary judgments:* An *iid* sequence of random variables $X_1, \cdots, X_T$ with domains $\mathcal{D}_{X_1}, \cdots, \mathcal{D}_{X_T}$ generates T elementary judgments $x_1, \cdots, x_T$ progressively expressed along T steps.

2. *Accumulating function:* At each step t, a function $f : \mathcal{D}_{X_1} \times \cdots \times \mathcal{D}_{X_t} \to \Psi_t \subseteq \mathbb{R}$ summarizes the t past elementary judgments (for example, by summation). We say that f is an accumulating function, i.e. we require it obeys the following property: $\Psi_t \subseteq \Psi_{t+1}$, $\forall t$.

3. *Accumulated judgments:* A sequence of random variables $W_1, \cdots, W_T$, $W_t = f\left(X_1, \cdots, X_t\right)$, with domains $\mathcal{D}_{W_1} \equiv \Psi_1, \cdots, \mathcal{D}_{W_T} \equiv \Psi_T$ is then originated along the T steps of the DP with T corresponding realizations $w_1, \cdots, w_T$, $w_t = f\left(x_1, \cdots, x_t\right)$, called accumulated judgments.

4. *'Likertization' function:* At each step t, a non-decreasing function $d : \mathcal{D}_{W_T} \to (1, \cdots, m)$ transforms w_t into a provisional rating. Note that from the definition of accumulating function derives $\mathcal{D}_{W_1} \subseteq \cdots \subseteq \mathcal{D}_{W_T}$, so that d can always be computed on the domain of W_t, for all t.

5. *Provisional ratings:* A sequence of random variables $R_1, \cdots, R_T$, $R_t = d(W_t)$, with domains the space $(1, \cdots, m)$ is then originated along the T steps of the feeling path with T corresponding realizations $r_1, \cdots, r_T$, $r_t = d(w_t)$, called provisional ratings.

B) Uncertainty Approach

Uncertainty judgment: A value q, called uncertainty judgment, is drawn by a random variable Q with domain $\mathcal{D}_Q \equiv (1, \cdots, m)$, $Q \perp R_T$, where $\perp$ denotes stochastic independence.

C) Expressed Rating

Final rating: The composition of the feeling and uncertainty components of the DP is formalized by a random variable R, with domain the space $(1, \cdots, m)$ and distribution given by a mixture of R_T and Q with weight π. The expressed rating r deriving from this DP is a realization of R.

Since the described process allows us to model the elementary judgments both using discrete and continuous random variables X_t, we will talk of a discrete and a continuous DP, respectively.

The proposed general DP comprises several statistical models and techniques already existing in the literature. For example, when $\pi = 1$, W_T is a continuous random variable (for example, normally distributed) obtained by accumulating elementary judgements by some f, and d is a generic 'Likertization' function with unknown parameters, we find the usual framework for latent variables very common in the literature. In fact, standard techniques dealing with ordinal data consider a continuous latent variable that is partially observed through the ordered categories, which correspond to adjacent intervals defined by cutpoints; this approach is implicit in the proportional odds model (McCullagh, 1980) and in other generalized linear models (McCullagh and Nelder, 1989; Tutz, 2012; Agresti, 2013). Additionally, the well-known Structural Equation Modelling was developed based on the assumption that each multivariate observed variable has an underlying continuous latent variable (Bentler, 1980).

Hereafter, we will limit ourselves to consider DPs belonging to the subset defined by the following assumptions: a discrete DP (H1) and an additive accumulating function f, i.e. $W_t = X_1 + \cdots + X_t$ (H2). Note that the function assumed in H2 meets the requirement of accumulating function, as defined at point A2 of the above scheme ($\Psi_t \subseteq \Psi_{t+1}$, $\forall t$) if and only if the domains of the *iid* random variables $X_1, \cdots, X_T$ contain the neutral element with respect to addition, i.e. 0. Thus, assuming an additive accumulating function f implies this former assumption for the domains $\mathcal{D}_{X_1}, \cdots, \mathcal{D}_{X_T}$. Hereafter, the DPs belonging to the subset defined by H1-H2 will be called 'discrete additive'.

3.1 Transition probabilities of discrete additive DPs

An interesting feature of the described DP is the probability of an increase of one rating point in the next step of the feeling path of the DP. Formally, we denote with $\phi_t(s) = Pr(R_{t+1} = s + 1 | R_t = s)$ the probability of moving to the provisional rating $r_{t+1} = s + 1$ at the $(t + 1)$-th step of the process, given that the provisional rating at step t is $r_t = s$. This probability is defined for $t : Pr(R_t = s) \neq 0$, $t < T$, and for $s < m$. In general, $\phi_t(s)$ can be obtained as the probability that the basic judgment x_{t+1} assumes a value allowing $d(w_{t+1}) = s + 1$, given that the basic judgments accumulated up to step t, w_t, are such that $d(w_t) = s$. Following this description, for a discrete additive DP we obtain

$$\phi_t(s) = \frac{\sum_{w_t \in d^{-1}(s)} Pr(\underline{x}(s) < X_{t+1} \leq \overline{x}(s) | W_t = w_t) Pr(W_t = w_t)}{\sum_{w_t \in d^{-1}(s)} Pr(W_t = w_t)}, \tag{1}$$

with $t : \mathcal{D}_{W_t} \cap d^{-1}(s) \neq \emptyset$, $t < T$, where $\underline{x}(s) = \max\{d^{-1}(s)\} - w_t$ and $\overline{x}(s) = \max\{d^{-1}(s + 1)\} - w_t$. In order to consider what happens during the first step of the DP, we define $w_0 := 0$ and $\phi_0 = \phi_0(s) := Pr(\underline{x}(s) < X_1 \leq \overline{x}(s))$ with $s = d(w_0) = d(0)$. The conditional probabilities $\phi_t(s)$ are able to reveal the structure of the respondents' perceptions about the rating scale, and the way, for what is concerned with the feeling component of the DP, they generate the final rating in their mind. In general, it can be useful to consider the average transition probabilities

$$\phi(s) = av_t(\phi_t(s)), \tag{2}$$

where av_t denotes averaging over t. Note that the value $\phi(s)$ can be interpreted as a 'perceived closeness' between ratings s and $s + 1$. In fact, a high value of $\phi(s)$ indicates that the respondents find it easy to move from rating s to $s + 1$, and this means that there is a small gap between the two ratings in the

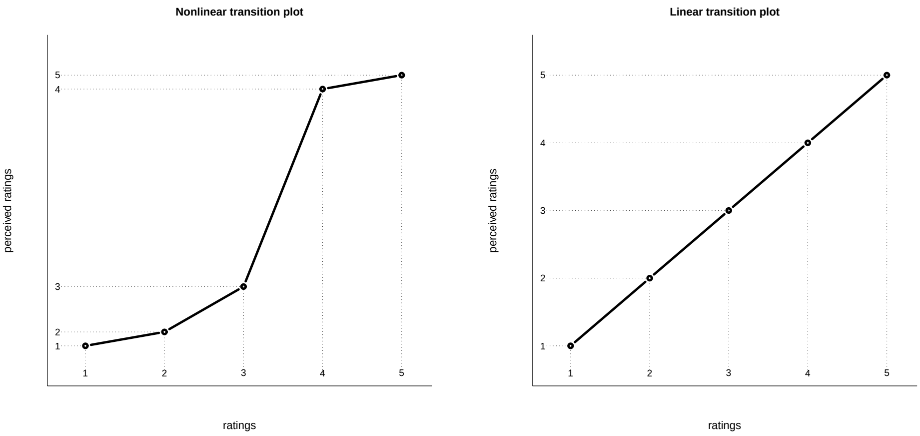

Figure 1: Examples of transition plot

respondents' perceptions. Conversely, a low value of $\phi(s)$ indicates that the respondents find it hard to move from rating s to $s+1$, and this indicates a large gap between the two ratings in the respondents' perceptions. For ease of interpretation, we can define a 'perceived distance' function between rating s and $s+1$, $\delta_s = h(\phi(s))$. For example, we could set $\delta_s = -\log(\phi(s))$ or $\delta_s = 1 - \phi(s)$. Therefore, transition probabilities account for a possible non-equally spacing of perceived ratings. This can be visualized by means of a graph, which we call *transition plot*, where the points $(s, \tilde{\phi}(s-1))$, $s = 1, \cdots, m$, $\tilde{\phi}(0) = 0$, and $\tilde{\phi}(s-1) = (\delta_1 + \cdots + \delta_{s-1})/(\delta_1 + \cdots + \delta_{m-1})$ for $s = 2, \cdots, m$, are joined by a broken line. By construction, the y-axis in the transition plot ranges in the [0,1] interval.

Two examples of transition plot are reported in Figure 1. Looking at the graph in the left panel, we notice that the slope of the transition plot between ratings 3 and 4 is higher than, for example, between ratings 2 and 3. This means that ratings 3 and 4 are more spaced in the perceptions of respondents. Therefore, the Likert-scaled ratings $1, \cdots, m$ are represented by the equally-spaced points on the x-axis, and the corresponding points on the y-axis can be interpreted as the perceived ratings, unequally-spaced in the perceptions of respondents. Of course, we can draw a transition plot for each step t of the feeling path, using the transition probabilites $\phi_t(s)$ in place of $\phi(s)$.

We can also define the expected number of one-rating-point increments during the feeling path

$$\mu = \phi_0 + \sum_{t=1}^{T-1} \sum_{s=1}^{m-1} \phi_t(s) Pr(R_t = s) \tag{3}$$

and the unconditional probability of increasing one rating point in one step of the feeling path

$$\phi = E_t(Pr(R_{t+1} = R_t + 1)) = \frac{1}{T} \left[\phi_0 + \sum_{t=1}^{T-1} \sum_{s=1}^{m-1} \phi_t(s) Pr(R_t = s) \right] = \frac{\mu}{T} \quad . \tag{4}$$

Definition: A DP is said to be linear if $\phi_{t_1}(s_1) = \phi_{t_2}(s_2) = \phi$, $\forall s_1, s_2, t_1, t_2 : \exists\ \phi_{t_1}(s_1), \phi_{t_2}(s_2)$.

From this definition, the linearity of the DP is concerned with a constant probability of increasing one rating point in the next step of the DP, regardless both the step and the rating already reached. Note that for a linear DP the transition plots are all linear, like the right plot in Figure 1, indicating that the ratings are equally spaced in the respondents' perceptions. For a linear DP, we also have $\phi_t(s) = \phi$ for all t, s.

4 Nonlinear CUB models

In this section, we propose a new model to fit rating data, derived as a special case of the DP proposed in Section 3, which is called Nonlinear CUB model (NLCUB) because it is designed as a generalization of the CUB class, aimed at modelling also nonlinear DPs.

Let us now recall Example 2, which intuitively introduced the underlying unconscious DP characterizing the NLCUB models. The DP of Example 2 fits into the general framework formalized in Section 3, with $T = 8$, $m = 5$, $X_1, \cdots, X_8$ following a Bernoulli distribution, $f(X_1, \cdots, X_t) = X_1 + \cdots + X_t$, $d(w)$ given by

$$d(w) = \begin{cases} 1 & \text{if} \quad w = 0 \\ 2 & \text{if} \quad w \in [1, 2] \\ 3 & \text{if} \quad w \in [3, 4, 5, 6] \\ 4 & \text{if} \quad w = 7 \\ 5 & \text{if} \quad w = 8 \end{cases} \tag{5}$$

and Q drawn from a Uniform distribution with domain $(1, \cdots, 5)$. We easily note that the NLCUB models are defined with exactly the same features of the CUB models, except for the number of steps of the feeling path and the 'Likertization' function d. In detail, the NLCUB models have the structure of a discrete DP with $T > m - 1$ and the features listed below:

A) Feeling Approach

1. *Elementary judgments:* The variables X_t, $t = 1, \cdots, T$, follow a Bernoulli distribution with success parameter $1 - \xi$, $\xi \in [0, 1]$, $X_t \sim B(1 - \xi)$.
2. *Accumulating function:* The accumulating function f is additive, i.e. $W_t = X_1 + \cdots + X_t$.
3. *Accumulated judgments:* As a consequence, the variables W_t, $t = 1, \cdots, T$, follow a Binomial distribution with parameters t and $1 - \xi$, $W_t \sim Bin(t, 1 - \xi)$.
4. *'Likertization' function:* The 'Likertization' function d creates a mapping from the domain of W_T, $\mathcal{D}_{W_T} = (0, \cdots, T)$, to $(1, \cdots, m)$. Note that $|(0, \cdots, T)| > |(1, \cdots, m)|$, where $| \cdot |$ denotes the cardinality of a set, so $d^{-1}(s)$ has necessarily to contain two or more values for some rating s, $s = 1, \cdots, m$. Many different functions d are allowed.
5. *Provisional ratings:* The random variables R_t, $t = 1, \cdots, T$, generating the provisional ratings follow a distribution depending on d.

B) Uncertainty Approach

Uncertainty judgment: The random variable Q follows a Uniform distribution with domain the discrete space $(1, \cdots, m)$.

C) Expressed Rating

Final rating: The variable R generating the final rating is then a mixture of R_T and Q, with weight π, $\pi \in (0, 1]$, and with domain $(1, \cdots, m)$.

This model belongs to the subset of discrete additive DPs and, as we will see later, is able to describe nonlinear DPs. The 'Likertization' function d plays an essential role in determining the nonlinearity pattern of the DP. In fact, it is d that, being applied to the domain of the accumulated judgments in order to obtain the provisional ratings, determines how many favourable elementary judgments are needed to reach a given rating s. With the classical CUB models, each additional favourable elementary judgment results in the increase of one rating point. With the NLCUB models, in order to increase one rating point, we may need to accumulate a higher number of favourable elementary judgments, and this number can be different according to the rating being increased. In this context, the 'Likertization' function d is defined as

$$d(w) = \begin{cases} 1 & \text{if} \quad w \in [w_{11}, \ldots, w_{g_1 1}] \\ 2 & \text{if} \quad w \in [w_{12}, \ldots, w_{g_2 2}] \\ \vdots & \vdots \quad \vdots \\ m & \text{if} \quad w \in [w_{1m}, \ldots, w_{g_m m}] \end{cases}, \tag{6}$$

where w_{hs} is the h-th element of $d^{-1}(s)$. Note that $(w_{11}, \ldots, w_{g_1 1}, w_{12}, \ldots, w_{g_2 2}, \ldots, w_{1m}, \ldots, w_{g_m m}) = (0, \cdots, T) = \mathcal{D}_{W_T}$. We denote with $g_s = |d^{-1}(s)|$ the number of values of $\mathcal{D}_{W_T}$ to which rating s corresponds based on d. The values $g_1, \cdots, g_m$ univocally determine the function d and can be considered parameters of the model. We have $T = g_1 + \cdots + g_m - 1$. If we lighten the condition $T > m - 1$ by allowing

$T \geq m-1$, this framework includes the classical CUB models as a particular case, when $T = m-1$ and $g_s = 1$ for all s, $s = 1, \cdots, m$.

Since the NLCUB models derive from a discrete additive DP, the transition probabilities $\phi_t(s)$ can be obtained by (1). By taking into account the Bernoulli distribution of variables X_t and the Binomial distribution of variables W_t, we easily obtain

$$\phi_t(s) = (1-\xi) \frac{\binom{t}{w_{g_s s}} (1-\xi)^{w_{g_s s}} \xi^{t-w_{g_s s}}}{\sum_{h=1}^{g_s} \binom{t}{w_{hs}} (1-\xi)^{w_{hs}} \xi^{t-w_{hs}}}. \quad (7)$$

Since $d(0) = 1$, $\phi_0 = \phi_0(1) = 0$ if $g_1 > 1$ and $\phi_0 = \phi_0(1) = 1-\xi$ if $g_1 = 1$. Here the condition $t : \mathcal{D}_{W_t} \cap d^{-1}(s) \neq \emptyset$, $t < T$, can be better specified as $w_{1s} \leq t < T$.

The expected number of one-rating-point increments in the feeling path is given by

$$\mu = \phi_0 + (1-\xi) \sum_{t=1}^{T-1} \sum_{s=1}^{m-1} \binom{t}{w_{g_s s}} (1-\xi)^{w_{g_s s}} \xi^{t-w_{g_s s}} \quad (8)$$

and we have $1 + \mu = E(R_T)$.

5 Real data analysis

In this section, we present a case study dealing with data from the Standard Eurobarometer 78, a sample survey covering the national population of citizens of the 27 European Union member states. The number of interviewees ranges from 500 (Malta) to 1,561 (Germany), with an average of 986 over the 27 considered countries. More details can be found on the technical notes and the annexes to the report on Standard Eurobarometer 78 (Eurobarometer 78.1 (2012): TNS Opinion & Social, Brussels, available from http://ec.europa.eu/public_opinion/archives/eb/eb78/eb78_en.htm). In this section, we aim to shed light on the insights allowed by NLCUB and its advantages over the standard CUB models, but not to draw economic conclusions. For this reason, and for the sake of brevity, we will present results of only one question (QA3.2: 'How would you judge the current situation of the European economy?') and three selected countries, which exhibit behaviours that we judge interesting from a statistical point of view (Greece, Germany, Italy). The ratings were expressed on a Likert scale with $m = 4$ possible responses ('very bad', 'rather bad', 'rather good', 'very good'). The 'don't know' option has been treated with listwise deletion (see Manisera and Zuccolotto, 2014, for an application on the same data taking account of the 'don't know' responses).

We fitted data with standard CUB and NLCUB models, with $T_{max} = 2m-1 = 7$. Figure 2 shows the observed relative frequencies and the corresponding (CUB and NLCUB) fitted probabilities for Greece, Germany and Italy, while parameter estimates are reported in Table 3.

In the case of Greece, the CUB model almost perfectly fits the observed frequencies. This evidence is confirmed, as the estimated NLCUB exactly matches the CUB structure ($g_s = 1$, $s = 1, \cdots, 4$). On the other hand, for Germany and Italy, the NLCUB estimates of $g_1, \cdots, g_4$ suggest a nonlinear structure, which appreciably improves the fit of CUB. The observed frequency distributions are characterized by a high value on the second response category ('rather bad'), but it cannot be modelled as a shelter effect, because with $m = 4$ the Shelter CUB model is saturated (Iannario and Piccolo, 2014). In both cases, CUB and NLCUB models provide similar estimates for the feeling and uncertainty parameters μ and $1-\pi$, but the CUB model is not able to reveal the non-constantness of the transition probabilities, which instead decrease when moving towards the higher ratings.

On the whole, all respondents from the three countries exhibit a very low uncertainty, while some differences can be observed from the feeling point of view. In fact, the parameters μ reveal that Italian respondents are the most pessimistic about the European economy, immediately followed by Greek respondents. German people are more confident. Figure 3 shows the transition plots implied by the estimated NLCUB models for the three countries. Unlike Greece, which, as expected, shows a linear transition plot, Germany and Italy show two slightly different nonlinear DPs, both characterized by a decreasing probability of moving to higher ratings. This means that, in general, for Greek respondents moving, for instance, from rating 1 to 2 is as hard as moving from rating 3 to 4. German and Italian respondents find it easier moving from rating 1 to 2 than moving from rating 3 to 4.

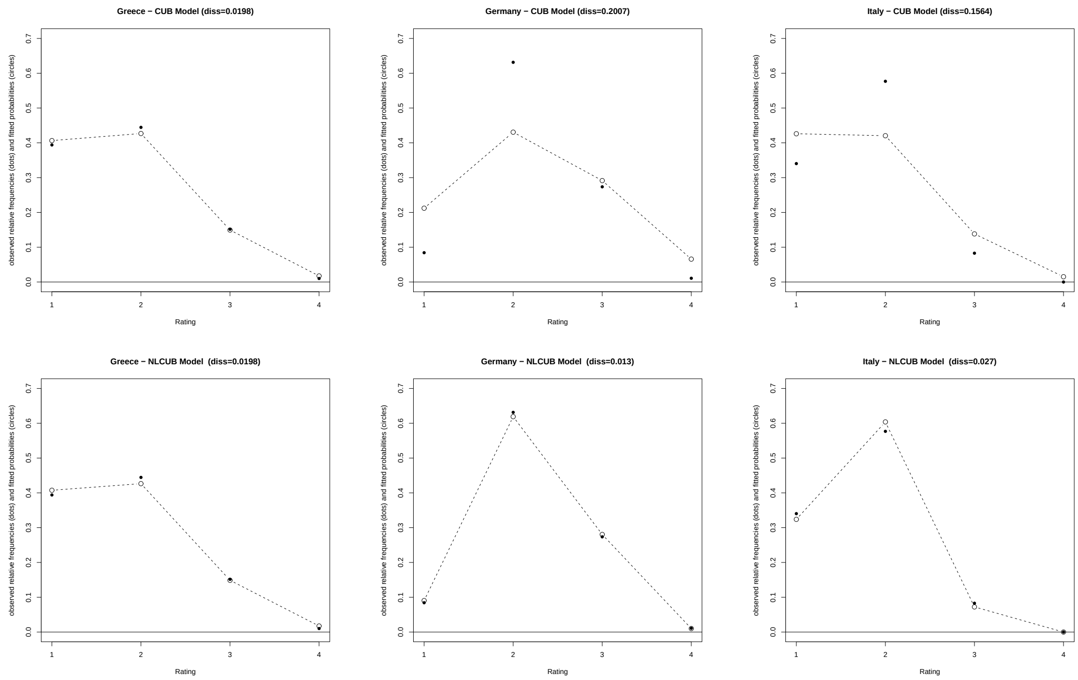

Figure 2: Observed relative frequencies vs fitted probabilities (CUB and NLCUB), Standard Eurobarometer 78 (QA3.2), Greece (left), Germany (middle), Italy (right)

Table 3: CUB and NLCUB parameter estimates, Standard Eurobarometer 78 (QA3.2), Greece, Germany and Italy

Greece	ξ	π	g_1	g_2	g_3	g_4
NLCUB	0.7413	0.9999	1	1	1	1
CUB	0.7413	0.9999	1*	1*	1*	1*
		$\phi(1)$	$\phi(2)$	$\phi(3)$	ϕ	μ
NLCUB		0.2587	0.2587	0.2587	0.2587	0.7761
CUB		0.2587	0.2587	0.2587	0.2587	0.7761

Germany	ξ	π	g_1	g_2	g_3	g_4
NLCUB	0.6167	0.9925	1	2	2	1
CUB	0.5964	0.9999	1*	1*	1*	1*
		$\phi(1)$	$\phi(2)$	$\phi(3)$	ϕ	μ
NLCUB		0.3833	0.1057	0.0258	0.2416	1.2080
CUB		0.4036	0.4036	0.4036	0.4036	1.2108

Italy	ξ	π	g_1	g_2	g_3	g_4
NLCUB	0.8512	0.9999	1	2	4	1
CUB	0.7525	0.9999	1*	1*	1*	1*
		$\phi(1)$	$\phi(2)$	$\phi(3)$	ϕ	μ
NLCUB		0.1487	0.0249	8.7e-06	0.1069	0.7484
CUB		0.2475	0.2475	0.2475	0.2475	0.7425

* Values of $g_1, \cdots g_4$ for CUB models are fixed

6 Conclusions

In this paper, we started from a general model for the cognitive process that leads a person to express a Likert-scaled rating about some latent trait. This model assumes the presence of two different compo-

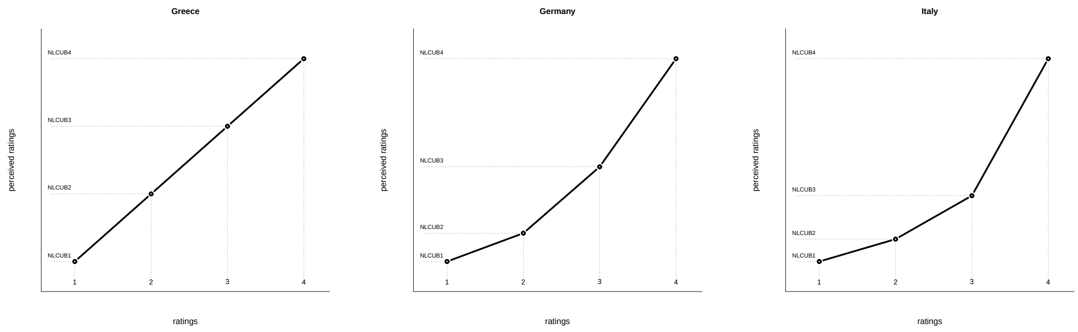

Figure 3: Transition plots of the NLCUB models, Standard Eurobarometer 78 (QA3.2), Greece (left), Germany (middle), Italy (right)

nents in the decision process, which, borrowing the terminology of the CUB models, we called feeling and uncertainty components, respectively. The former accounts for logic reasoning as well as emotions and perceptions, while the latter embodies other elements not generally directly connected with the latent trait being evaluated. On one hand, with a given probability, the final rating is determined within the feeling approach, with a decision path proceeding through a number of consecutive steps, so that the last feeling rating is the result of many elementary judgments that are, firstly, summarized and, secondly, transformed into a Likert-scaled rating. On the other hand, the final rating can result from the uncertainty approach, consisting of a random judgment independent on the feeling path. The main feature of the proposed model is the possibility of expressing the so-called transition probability, i.e. the probability of increasing one rating point at a given step of the feeling path in the decision process. Transition probabilities and the related transition plots are able to describe the state of mind of the respondents about the response scale used to express judgments. We have shown that this general framework can include some well-known models for ratings, by changing the assumptions about the probability distributions of the random variables involved in the process or about the functions used to summarize the elementary judgments and transform them into Likert-scaled ratings. In particular, we have shown that the proposed general framework includes the popular class of the CUB models, that, after being introduced in Piccolo (2003), have earned a wide interest in the literature. After having recognized the way the CUB models comply with the general framework, we have been able to generalize their structure and introduce a new class of models called Nonlinear CUB, which include CUB as a particular case. We have shown that, unlike the classical CUB models, the Nonlinear CUB models are able to fit decision processes with non-constant transition probabilities. A real data case study has shown that this feature provides interesting insights on the analysed phenomenon.

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