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Abstract

In high-dimensional vector autoregressive (VAR) models, it is natural to have large number of predictors relative to the number of observations, and a lack of efficiency in estimation and forecasting. In this context, model selection is a difficult issue and standard procedures may often be inefficient. In this paper we aim to provide a solution to these problems. We introduce sparsity on the structure of temporal dependence of a graphical VAR and develop an efficient model selection approach. We follow a Bayesian approach and introduce prior restrictions to control the maximal number of explanatory variables for VAR models. We discuss the joint inference of the temporal dependence, the maximum lag order and the parameters of the model, and provide an efficient Markov chain Monte Carlo procedure. The efficiency of the proposed approach is showed on simulated experiments and real data to model and forecast selected US macroeconomic variables with many predictors.

Keywords: High-dimensional Models, Large Vector Autoregression, Model Selection, Prior Distribution, Sparse Graphical Models, Bayesian Vector Autoregressive Models

JEL: C11, C15, C52, C55, E17

1. Introduction

High dimensional modeling and large dataset handling have recently gain attention in several fields, particularly in economics and finance. This has become necessary since useful information is often scattered among large number of variables. Building models that allow for extraction of these information from large dataset enhances a better understanding of the modern economic and financial system. Many researchers have shown that combining financial and macroeconomic variables to estimate large vector autoregressive (VAR) models produces better forecasts than standard approaches (see, *Address: Department of Economics, Ca’ Foscari University of Venice, Fondamenta San Giobbe 873/b, 30121, Venice, Italy. Corresponding author: Daniel Felix Ahelegbey, dfkahey@unive.it. Other contacts: billio@unive.it (Monica Billio); r.casarin@unive.it (Roberto Casarin).

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Banbura et al., 2010; Carriero et al., 2013; Giannone et al., 2005; Koop, 2013; Stock and Watson, 2012). Many others, using datasets of a large number of financial institutions, have shown that the financial system has become highly interconnected and thus, can be represented as a network where linkages among agents sharing common structures play a fundamental role in contagion and the spread of systemic risk (see, Billio et al., 2012; DasGupta and Kaligounder, 2014; Diebold and Yilmaz, 2014; Hautsch et al., 2014; Huang et al., 2012).

In this paper we propose a new Bayesian model for multivariate time series of large dimension by combining graph-based notion of causality (see Lauritzen and Wermuth, 1989; Pearl, 1988; Whittaker, 1990), with the concept of sparsity (see, e.g. Box and Meyer, 1986). Graphical models have been applied in time series analysis for estimating causal structures in VAR models (see Corander and Villani, 2006; Demiralp and Hoover, 2003; Moneta, 2008; Swanson and Granger, 1997) and identification restrictions in structural VAR (Ahelegbey et al., 2015). They have received increasing attention as tools to represent interconnectedness and sources of contagion among financial institutions (see Ahelegbey and Giudici, 2014; Barigozzi and Brownlees, 2014; Billio et al., 2012; Diebold and Yilmaz, 2014; Merton et al., 2013). As described in the following, we contribute to the literature in many ways.

One of the key challenges of high-dimensional models is the complex interactions among variables and the inferential difficulty associated with handling large datasets. For instance, in large VAR models, econometricians encounter the curse of dimensionality problem due to high number of variables relative to the number of data points. The standard Bayesian VAR approach to this problem is to apply Minnesota prior by Doan et al. (1984), as a solution to overfitting. This approach is however inefficient to deal with the problem of indeterminacy (see Donoho, 2006), i.e. when the number of parameters in a system of equations exceeds the number of observations. Two common approaches to the indeterminacy issue discussed in the literature are based alternatively on dimension reduction or variable selection methodologies. For dimension reduction, dynamic factor models, factor augmented VAR and Bayesian model averaging have been extensively discussed and widely considered to extract useful information from a large number of predictors (see Bai and Ng, 2008; Bernanke et al., 2005; Geweke, 1977; Giannone et al., 2005; Koop and Potter, 2004; Stock and Watson, 2006). For variable selection, standard techniques have been applied to reduce the number of predictors, e.g., the Least Absolute Shrinkage and Selection Operator (LASSO) of Tibshirani (1996), and its variants, (see, e.g. Efron et al., 2004; Kyung et al., 2010; Park and Casella, 2008; Zou and Hastie, 2005). The method considered in this paper is related to the latter, thus to variable selection.

Variable selection is a fundamental problem in high-dimensional models, and this
is closely related to the possibility to describe the model with sparsity (Zhang et al., 2012). The idea of sparsity is associated with the notion that a large variation in the dependent variables is explained by a small proportion of predictors (Box and Meyer, 1986). Modeling sparsity has received attention in recent years in many fields, including econometrics, (see Elliott et al., 2013; Gefang, 2014; Korobilis, 2013). See also Belloni and Chernozhukov (2011) for an introduction to high-dimensional sparse econometric models.

This paper introduces and models sparsity in graphical VAR models of large dimension by dealing also with uncertainty in the lag order. It thus substantially extends the graphical VAR model, the inference approach and posterior approximation algorithm given in Ahelegbey et al. (2015). In most empirical analyses, researchers often overlook dependence among series when dealing with multi-equation regression models and large number of predictors, (see, e.g. Korobilis, 2013; Stock and Watson, 2014), since model selection is a difficult issue and such approach is often necessary to avoid the indeterminacy problem. However, this can be unsatisfactory in terms of interpretability and forecasting performance, since temporal dependence in the series is ignored. The graphical approach presented in this paper enables us to deal with this indeterminacy problem by exploiting sparsity to estimate the dynamic causal structure in large VAR models.

Many studies have considered several approaches to model sparse graphs (see, e.g. Carvalho et al., 2008; Jones et al., 2005; Scott and Carvalho, 2008; Shojaie and Michailidis, 2010). Also, there is an increasing interest in sparsity estimation for large VAR models (see, e.g. Davis et al., 2012; De Mol et al., 2008; Gefang, 2014; Kock and Callot, 2012; Medeiros and Mendes, 2012; Song and Bickel, 2011). We contribute to this literature by focusing on graphical VAR models from a Bayesian perspective with suitable prior specifications to deal with sparsity on the temporal dependence. More precisely, we build on the fan-in method of Friedman and Koller (2003) and propose a new approach to sparsity modeling. The idea of the fan-in is based on imposing a maximal number of predictors to ensure sparsity on the graph. Setting an a-priori hard fan-in might be too restrictive for large VAR applications. We therefore propose a prior distribution on the fan-in to allow for different prior information level about the maximal number of predictors for each equation of the VAR model. Thus, we allow for a random fan-in and adapt this prior distribution to the prior probability in variable selection problems. We show that this new prior distribution encourages sparsity on the graph taking into account the lag order. Since there is duality between prior and the penalty in the information criterion, our prior leads to a modified BIC for graphical model selection.

We also contribute to the literature on dynamic relationship identification. Here, we propose an efficient Markov Chain Monte Carlo (MCMC) algorithm to sample jointly,
the graph structure, the lag order and the parameters of the VAR model. Due to the uncertainty on the lag order, we propose an efficient MCMC algorithm that takes advantage of computational power through parallel simulation of the graph and lag order. Inference of the graph and lag order allows us to estimate only the parameters of the relevant explanatory variables.

We show the efficiency and study the performance of our approach through simulation examples and an application to forecast macroeconomic times series with large number of predictors. We consider the standard Lasso-type methods (i.e. LASSO and Elastic-Net) as benchmarks for comparing the identified causal structure and the forecast ability. We find evidence that our sparse graphical VAR model is more parsimonious than the LASSO and Elastic-Net. Furthermore, we find evidence of a gain in the predictive accuracy of our approach over the Lasso-type methods.

The paper is organized as follows: Section 2 presents the graphical concept for VAR models; Section 3 discusses prior distributions and focuses on the interaction between lag order and sparse graph prior distribution; Section 4 discusses the Bayesian inference scheme; Section 5 illustrates the simulation experiments; and Section 6 presents the empirical application.

2. Graphical VAR Models

Graphical models are statistical models that summarize the marginal and conditional independences among random variables by means of graphs (see Brillinger, 1996). The core of such models is representing relationships among variables in the form of graphs using nodes and edges, where nodes denote variables and edges show interactions. They can be represented in a more compact form by \((G, \theta) \in (\mathcal{G} \times \Theta)\), where \(G\) is a graph of relationships among variables, \(\theta\) is the graphical model parameters, \(\mathcal{G}\) is the space of the graphs and \(\Theta\) is the parameter space.

Let \(X_t\) be a \(n \times 1\) vector of observations at time \(t\) and assume \(X_t = (Y_t', Z_t')\), where \(Y_t\), the \(n_y \times 1\) vector of endogenous variables, and \(Z_t\), a \(n_z \times 1\), \(n_z = (n - n_y)\) vector of exogenous predictors. In a VAR model with exogenous variables, the variables of interest \(Y_t\), is determined by the equation

\[
Y_t = \sum_{i=1}^{p} B_i X_{t-i} + \varepsilon_t
\]

\(t = 1, \ldots, T\), where \(\varepsilon_t\) is a \(n_y \times 1\) vector of errors, independent and identically normal, with mean zero and covariance matrix \(\Sigma_{\varepsilon}\); \(p\) is the maximum lag order; \(B_i\), \(1 \leq i \leq p\) is a \(n_y \times n\) matrix of coefficients.
By interpreting (1) as a model with temporal dependence between explanatory and dependent variables, the VAR model can be expressed in a graphical framework (referred to as graphical VAR model), with a one-to-one correspondence between the coefficient matrices and a directed acyclic graph; if \( B_{s,ij} \neq 0 \) then there is a causal effect of \( X_{t-s}^i \) on \( Y_t^j \), \( 1 \leq s \leq p \). Here we read \( X_{t-s}^i \) as realization of the \( i \)-th element of \( X \) at time \( t \).

More formally, we define the relation \( B_s = (G_s \circ \Phi_s) \), where \( G_s \) is a \( n_y \times n \) binary connectivity matrix (also called adjacency matrix), \( \Phi_s \) is a \( n_y \times n \) coefficients matrix, and the operator \( \circ \) is the element-by-element Hadamard’s product (i.e., \( B_{s,ij} = G_{s,ij} \Phi_{s,ij} \)). Based on this definition, we identify a one-to-one correspondence between \( B_s \) and \( \Phi_s \) conditional on \( G_s \), such that \( B_{s,ij} = \Phi_{s,ij} \) if \( G_{s,ij} = 1 \); and \( B_{s,ij} = 0 \) if \( G_{s,ij} = 0 \). The above relationship can be presented in a stacked matrix form as, \( B = (G \circ \Phi) \), where \( B = (B_1, \ldots, B_p) \), \( G = (G_1, \ldots, G_p) \) and \( \Phi = (\Phi_1, \ldots, \Phi_p) \), where each matrix is of dimension \( n_y \times np \).

Let \( W_t \) be stacked lags of \( X_t \), where \( W_t = (X_{t-1}^1, \ldots, X_{t-p}^1)' \) is of dimension \( np \times 1 \), with \( p \) as the lag order, and \( V_t = (Y_t^1, W_t^1)' \) of dimension \( (n_y + np) \times 1 \). Suppose the joint, \( V_t \), follows the distribution, \( V_t \sim \mathcal{N}(0, \Sigma^{-1}) \), where \( \Sigma = \Omega^{-1} \) is \( (n_y + np) \times (n_y + np) \) is the covariance matrix. The joint distribution of the variables in \( V_t \) can be summarized with a graphical model, \((G, \theta)\), where \( G \in \mathcal{G} \) consists of directed edges. In this paper, we focus on modeling directed edges from elements in \( W_t \) to elements in \( Y_t \), capturing the temporal dependence among the observed variables. More specifically, \( G_{ij} = 0 \), means the \( i \)-th element of \( Y_t \) and \( j \)-th element of \( W_t \) are conditionally independent given the remaining variables in \( V_t \), and \( G_{ij} = 1 \) corresponds to a conditional dependence between the \( i \)-th and \( j \)-th elements of \( Y_t \) and \( W_t \) respectively given the remaining variables in \( V_t \). The graphical model parameter, \( \theta \in \Theta \), consist the coefficients, capturing the strength and sign of the temporal dependence relationship described by \( G \). Based on the above assumption, estimating the model parameters associated with \( G \) is equivalent to estimating the precision matrix, \( \Omega \), i.e \( \theta = \Omega \). The relationship between the parameters of the VAR, \( \{B, \Sigma_c\} \), and that of the graphical model, \( \Omega \), is as follows.

**Proposition 1.** Suppose the marginal distribution of \( W_t \sim \mathcal{N}(0, \Sigma_{ww}) \) and the conditional distribution of \( Y_t|W_t \sim \mathcal{N}(BW_t, \Sigma_c) \). There is a correspondence between \( \{B, \Sigma_c\} \) and \( \Omega \), such that given \( \Omega \), we obtain \( \Sigma = \Omega^{-1} \) and \( \{B, \Sigma_c\} \) can be estimated by

\[
\begin{align*}
B &= \Sigma_{yw} \Sigma_{ww}^{-1}, \\
\Sigma_c &= \Sigma_{yy} - \Sigma_{yw} \Sigma_{ww}^{-1} \Sigma_{wy}
\end{align*}
\] (2)

Also given \( \{B, \Sigma_c\} \) and \( \Sigma_{ww} \), the precision matrix \( \Omega = \Sigma^{-1} \) of \( (Y_t, W_t) \) is estimated by

\[
\Omega = \begin{pmatrix}
\Sigma_c^{-1} & -\Sigma_c^{-1} B \\
-B' \Sigma_c^{-1} & \Sigma_{ww}^{-1} + B' \Sigma_c^{-1} B
\end{pmatrix}
\] (3)
Proof. See Appendix A.1.

Following our definition, \( B = (G \circ \Phi) \) and the results of Proposition 1, we identify the relationship between \( \Omega \) and the dependence structure \( G \), through the sub-matrix \( (\Sigma^{-1}_\epsilon B) \) of \( \Omega \). We denote \( \Omega^G = \Sigma^{-1}_\epsilon B \), defined on the space \( \mathcal{M}(G) \), i.e. the set of precision matrices where elements of \( \Omega^G \) corresponds to elements in \( G \). Clearly, if the errors are assumed to be independent and normally distributed, \( \Sigma_\epsilon \) is a diagonal matrix, which when normalized to identity leads to a one-to-one correspondence between \( B, \Omega^G \) and \( G \) such that \( B_{ij} = \Omega^G_{ij} = 0 \) if \( G_{ij} = 0 \) and \( B_{ij} = \Omega^G_{ij} \neq 0 \) if \( G_{ij} = 1 \). In large VAR models estimation, most empirical papers follow the above assumption on the errors to estimate the model, (see, e.g. Kock and Callot, 2012; Stock and Watson, 2014).

In this paper we assume \( \Sigma_\epsilon \) is a full matrix, i.e, the errors are correlated among the equations of the VAR. Estimating our graphical VAR model therefore involves: the temporal dependence graph, \( G \), the maximum lag order, \( p \), and the set of parameters in \( \Omega \) which related to \( \{B, \Sigma_\epsilon\} \). Estimating all these jointly is challenging. However, following the Bayesian paradigm allows us to take into account uncertainties on the quantities of interest and inference on these through model averaging, (Giudici and Green, 1999; Madigan and York, 1995). The objective of this paper is to estimate jointly the lag order and graph from the observed time series, and to incorporate the inferred quantities to select the relevant variables to estimate the parameters of the model.

3. Sparse Bayesian Graphical VAR Models

In a system of linear equations where the number of parameters exceeds the number of observations, for instance in large VAR models, we face another problem referred to as indeterminacy, (see Donoho, 2006). Such systems can be modeled by exploiting sparsity. The description of our graphical VAR for high dimensional multivariate time series is completed with the elicitation of the prior distributions for the lag order \( p \), a sparsity prior on the graph, and the prior on \( G \) and \( \Omega \).

3.1. Lag Order Prior Distribution

We allow for different lag orders for the different equations of the VAR model. We denote with \( p_i \) the lag order of the \( i \)-th equation. We assume for each \( p_i \), \( i = 1, \ldots, n_y \), a discrete uniform prior on the set \( \{\underline{p}, \ldots, \bar{p}\} \)

\[
P(p_i) = \frac{1}{(\bar{p} - \underline{p} + 1)} \mathbb{1}_{\{\underline{p}, \ldots, \bar{p}\}}(p_i) \tag{4}
\]
where \( l_A(x) \) is the indicator function, that is unity if \( x \in A \) and zero otherwise. This is a standard choice in AR model selection problems (e.g., see Casarin et al. (2012)). Alternatively, the lag order can be assumed to follow a truncated Poisson distribution with mean \( \lambda \) and maximum \( \bar{p} \) (see Vermaak et al. (2004)), or a discretized Laplace distribution (see Ehlers and Brooks (2004)). Our choice of discrete uniform distribution is fairly informative since \( p \) and \( \bar{p} \) are defined a-priori following standard applications and assigns equal weights to all possible values of \( p_i \). For instance, in estimating VAR models with monthly (quarterly) time series, the standard lag selection procedure often consider \( p = 1 \) and \( \ar{p} = 12 \) (\( \bar{p} = 4 \)). The alternative lag order prior distributions are more informative and assigns different weights to the possible values of \( p_i \).

### 3.2. Standard Graph Prior Distribution

Most of the literature on graphical models takes the prior for a graph \( G \) with \( n \) variables as uniform over all the relevant graphs, i.e., \( P(G) = |\mathcal{G}|^{-1} \), where \( |\mathcal{G}| \) is the cardinality of \( \mathcal{G} \), (see Geiger and Heckerman, 2002; Giudici and Castelo, 2003). This can be attributed to the fact that the number of possible graphs increases super-exponentially with the number of variables, and there is difficulty in calculating the number of possible graphs. Assuming uniform probabilities for all graphs, however, does not ensure sparsity. Thus, many authors have discussed several approaches to penalize globally or locally “dense” graphs (see, e.g. Carvalho et al., 2008; Jones et al., 2005; Scott and Carvalho, 2008; Wang, 2010). See also Telesca et al. (2012) and Shojaie and Michailidis (2009) for the use of explicit information prior to improve the estimation of the graph structure.

Friedman and Koller (2003) proposed a factorization of the graph prior into equation specific terms for DAG models. As argued by the authors, setting an upper bound on the number of explanatory variables for each dependent variable encourages sparsity on the graph. This bound is referred to as the fan-in restriction in the graphical model literature. Let \( m \) be the maximum number of explanatory variables for each equation. Restricting the graph model selection to at most \( f \) explanatory variables instead of \( m \), \( f < m \), reduces the number of possible sets from \( \mathcal{O}(2^m) \) to \( \binom{m}{f} \), where \( \binom{n}{k} \) is the binomial coefficient. A uniform choice on the latter set yields a graph prior

\[
P(G) = \prod_{i=1}^{n} P(\pi_i) \propto \prod_{i=1}^{n} \left( \frac{n - 1}{|\pi_i|} \right)^{-1}
\]

where \( \pi_i = \{ j = 1, \ldots, n : G_{ij} = 1 \} \) is the set of explanatory variables of the \( i \)-th equation, \( |\pi_i| \) is the number of elements in \( \pi_i \), and \( P(\pi_i) \) is the local graph prior of the \( i \)-th equation.
Jones et al. (2005) discussed a prior distribution for penalizing the inclusion of additional edges in dense graphs given by

\[ P(G|\gamma) = \gamma^k (1 - \gamma)^{m-k} \]  

(6)

where \( m \) is the maximum number of edges and \( k \) represents the number of edges in the graph. In their application, the authors use a Bernoulli prior on each edge inclusion with parameter \( \gamma = 2/(n - 1) \) and set \( m = \binom{n}{2} \).

For choices of the prior distribution on \( \gamma \) in the beta family, Scott and Carvalho (2008) showed that \( \gamma \) can be explicitly marginalized out. The uniform prior on \( \gamma \) gives a marginal prior inclusion probability of 1/2 for all edges and yields model probabilities

\[ P(G) = \frac{1}{(m+1)} \binom{m}{k}^{-1} \]  

(7)

3.3. Sparsity Prior Distribution

We build on the fan-in approach of Friedman and Koller (2003) by introducing a prior distribution on the fan-in to allow for different prior information level about the maximal number of predictors for each equation of the VAR model.

In a multivariate dynamic models with \( n \) variables and a lag order \( p \), the number of possible predictors is \( np \). Given that each series has \( T \) number of observations, then the number of observations of the model is \( T - p \). In large VAR models, it is often natural that the number of predictors is greater than the number of observations, i.e. \( np > T - p \). When this happens, we expect that each equation has at most \( T - p \) predictors to efficiently estimate the model. In cases where \( T - p > np \), we expect that each equation has at most \( np \) predictors. Thus the maximal number of explanatory variables required to efficiently estimate a high dimensional model is given by \( m_p = \min\{np, T - p\} \). Setting an a-priori hard fan-in (see Friedman and Koller, 2003) might be too restrictive for large VAR applications.

We denote with \( \bar{\eta}, 0 \leq \bar{\eta} \leq 1 \), the measure of the maximal density, i.e. the fraction of the predictors that explains the dependent variables. Thus the level of sparsity is given by \( (1 - \bar{\eta}) \). We set the fan-in to \( f = \lfloor \bar{\eta}m_p \rfloor \), where \( f \) is the largest integer less than \( \bar{\eta}m_p \). To allow for different levels of sparsity for the equations in the VAR model, we assume a prior distribution on the maximal density for the different equations. We denote \( \bar{\eta}_i \) the maximal density of the \( i \)-th equation and assume the prior on \( \bar{\eta}_i \), given lag order \( p_i \) is beta distributed with parameters \( a, b > 0 \), \( \bar{\eta}_i | p_i \sim \beta(a, b) \), on the interval \([0, 1]\)

\[ P(\bar{\eta}_i|p_i) = \frac{1}{B(a, b)} \bar{\eta}_i^{a-1}(1 - \bar{\eta}_i)^{b-1} \]  

(8)
This leads to random fan-in’s for the different equations in the VAR model. Note that the fan-in, \( f_i \), must be directly related to the number of selected predictors in each equation and indirectly related to the number of observations of the model.

3.4. Our Graph Prior Distribution

We define the graph prior for each equation in the VAR model conditional on the sparsity prior. We refer to the prior on the graph of each equation as the local graph prior, denoted by \( P(\pi_i|p_i, \gamma, \bar{\eta}_i) \). Following (Scott and Berger, 2010), we consider the inclusion of predictors in each equation as exchangeable Bernoulli trials with prior probability

\[
P(\pi_i|p_i, \gamma, \bar{\eta}_i) = \gamma^{||\pi_i||}(1 - \gamma)^{np_i - ||\pi_i||} I_{\{0, \ldots, f_i\}}(||\pi_i||)
\]

where \( \gamma \in (0, 1) \) is the Bernoulli parameter, \( ||\pi_i|| \) is the number of selected predictors out of \( np_i \) and \( f_i = \lfloor \bar{\eta}_i m_p \rfloor \) is the fan-in restriction for the \( i \)-th equation. We assign to each variable inclusion a prior probability, \( \gamma = 1/2 \), which is equivalent to assign the same prior probability to all models with predictors less than the fan-in \( f_i \), that is

\[
P(\pi_i|p_i, \bar{\eta}_i) = \frac{1}{2^{np_i}} I_{\{0, \ldots, f_i\}}(||\pi_i||)
\]

Alternatively, a uniform prior on \( \gamma \) gives to each variable a marginal prior inclusion probability of 1/2 and a local graph prior (Foygel and Drton, 2011) given by

\[
P(\pi_i|p_i, \bar{\eta}_i) = \left( \frac{np_i}{||\pi_i||} \right)^{-1} I_{\{0, \ldots, f_i\}}(||\pi_i||)
\]

3.5. Parameter Prior Distribution

There are two main approaches to define parameter priors for graphical models, however a common feature to these approaches is that both are graph conditional parameter priors. On one hand is a vast work on Gaussian DAG models discussing a list of conditions that permits an unconstrained precision matrix \( \Omega \) (see, e.g. Consonni and Rocca, 2012; Geiger and Heckerman, 2002; Heckerman, 1998; Heckerman and Chickering, 1995; Heckerman and Geiger, 1994). On the other hand is a vast publication on Gaussian decomposable undirected graphical (UG) models with constraints on the precision matrix \( \Omega \) (see, e.g. Carvalho and Scott, 2009; Lenkoski and Dobra, 2011; Roverato, 2002; Wang and Li, 2012). Note that, an unconstrained \( \Omega \) characterizes a complete Gaussian DAG or UG model, i.e. a graph with no missing edges. The standard parameter prior for Gaussian DAG models with zero expectations is a Wishart distribution, whereas that of UG models is a G-Wishart distribution. A consequence of the DAG conditional parameter prior is that, once we specify the parameter prior for one complete DAG model, all other
priors can be generated automatically (see Consonni and Rocca, 2012). In this paper, we follow the DAG model framework since it allows us to marginalize out $\Omega$ analytically thereby focusing on the structure inference, and the estimation of the model parameters given the structure of dependence (see Section 4 for details).

Following Geiger and Heckerman (2002), we assume a prior distribution on the unconstrained precision matrix, $\Omega$, conditional on any complete DAG, $G$, for a given lag order $p$, is Wishart distributed, with probability density function

$$P(\Omega|p, G) = \frac{1}{K_d(\nu, S_0)} |\Omega|^{(\nu-d-1)/2} \text{etr}\left(-\frac{1}{2} \Omega S_0\right)$$

where $\text{etr}(A) = \exp\{\text{tr}(A)\}$ and $\text{tr}(A)$ is the trace of a square matrix $A$, $\nu$ is the degree of freedom parameter, $S_0$ is a $d \times d$ symmetric positive definite matrix, with $d = n_y + np$, the size of the vector of stacked dependent and explanatory variables of the model. The normalizing constant is:

$$K_d(\nu, S_0) = 2^{d^2} |S_0|^{-\nu/2} \Gamma_d\left(\frac{\nu}{2}\right)$$

where $\Gamma_d(\cdot) = \pi^{d(d-1)/4} \prod_{i=1}^{d} \Gamma\left(\frac{1}{2} + \frac{1}{\nu} - \frac{i}{2}\right)$ is the multivariate gamma function and $\Gamma(\cdot)$ denotes the gamma function.

Based on the assumption that the conditional distribution of the dependent variables given the set of predictors, is described by equation (1), with parameters $\{B, \Sigma_\varepsilon\}$, we assume the prior distribution on $(B, \Sigma_\varepsilon|p, G)$ is an independent normal-Wishart (see, e.g. Geiger and Heckerman, 2002; Heckerman and Geiger, 1994). This is one of the prior distributions extensively applied in the Bayesian VAR literature. Specifically, we assumed the coefficients matrix $B$ is independent and normally distributed, $B|p, G \sim N(B, V)$, and $\Sigma_\varepsilon^{-1}$ is Wishart distributed, $\Sigma_\varepsilon^{-1} \sim W(\nu, S/\nu)$. The prior expectation, $B = 0_{n_y \times np}$, is a zero matrix, and the prior variance of the coefficient matrix, $V = I_{np \times np}$, is an identity matrix. Also, the prior expectation of $\Sigma_\varepsilon = \frac{1}{\nu} S$ where is $S$ is $n_y \times n_y$ positive semi-definite matrix and $\nu > n_y - 1$ is the degrees of freedom.

4. Bayesian Inference

We define $G_s$ as $n_y \times n$ binary connectivity matrix that captures the temporal relationship of variables at time $t - s$ with the variables at time $t$. We denote with $\tilde{G}_p = (G_1, \ldots, G_p)$ as stacked $G_1, \ldots, G_p$, such that $\tilde{G}_p$ is of dimension $n_y \times np$. We then define $\tilde{G}_{p,i}$, $i = 1, \ldots, n_y$ as the local graph associated with the $i$-th equation which is the $i$-th row of $\tilde{G}_p$. We proceed under the assumption that the series of dependent and explanatory variables is jointly Gaussian, $\mathcal{N}(0, \Omega^{-1})$. Moreover, conditional on the lag
order \( p \), any complete DAG, \( \tilde{G}_p \), and an unconstrained precision matrix \( \Omega \), the likelihood function is given by

\[
P(X|p, \tilde{G}_p, \Omega) = (2\pi)^{-\frac{m}{2}} |\Omega|^{-\frac{1}{2}} \text{etr}
\left(-\frac{1}{2} \Omega \hat{S} \right)
\]  

(14)

where \( \hat{S} = \sum_{i=1}^{T_0} V_i V_i' \), sum of squares matrix of dimension \( d \times d \). A nice feature of the unconstrained parameter prior in the DAG mode framework is that it allows for integrating out analytically, the precision matrix, \( \Omega \), with respect to its prior to obtain a marginal likelihood function for any DAG, \( \tilde{G}_p \) with lag \( p \) given by

\[
P(X|p, \tilde{G}_p) = \int P(X|p, \tilde{G}_p, \Omega) P(\Omega|p, \tilde{G}_p) \, d\Omega =
\frac{K_d(\nu + T_0, S_0 + \hat{S})}{(2\pi)^{\frac{m}{2}} K_d(\nu, S_0)}
\]  

(15)

where \( T_0 = T - p \), \( S_0 \) and \( S_0 + \hat{S} \) are the prior and posterior sum of square matrices, which when normalized are \( \Sigma = \frac{1}{\nu} S_0 \) and \( \bar{\Sigma} = \frac{1}{\nu + T_0} (S_0 + \hat{S}) \) respectively. Geiger and Heckerman (2002) outlined conditions for the integral in equation (15) to be analytically tractable and to have a close form expression that can be factorized into local marginal likelihoods. A key assumption is that the parameters must be independent within and across equations. In VAR models, the errors are correlated across equations which makes the factorization of (15) quite problematic. Following the idea of large-sample approximation by Kass et al. (1988) and Chickering and Heckerman (1997), we consider the errors of a large VAR model as unobserved variables which can be ignored when dealing with large datasets (see, e.g. Stock and Watson, 2014). Based on this consideration and the assumption that the coefficients in \( B \) are independent a-priori within and across equations, we approximate (15) with a pseudo-marginal likelihood given by the product of local densities

\[
P(X|p, \tilde{G}_p) \approx \prod_{i=1}^{n_y} P(X|p, \tilde{G}_p(y_i, \pi_i)) = \prod_{i=1}^{n_y} \frac{P(X^{(y_i, \pi_i)}|p_i, \tilde{G}_p)}{P(X^{(\pi_i)}|p_i, \tilde{G}_p)}
\]  

(16)

where \( \tilde{G}_p(y_i, \pi_i) \) is the local graph of the \( i \)-th equation with \( y_i \) as dependent variable and \( \pi_i \) as the set of predictors; \( X^{(y_i, \pi_i)} \) is the sub-matrix of \( X \) consisting of \( y_i \) and \( \pi_i \); and \( X^{(\pi_i)} \) is the sub-matrix of \( \pi_i \). This approximation allows us to develop search algorithms to focus on local graph estimation. More specifically, a Markov chain Monte Carlo (MCMC) algorithm using the global score would be less efficient in exploration since the global score would be insensitive to the proposal of edge deletion or addition. Thus, the approximation allows the chain to explore the graph locally at equation level. The pseudo-likelihood has been used in MCMC by Zhou and Schmidler (2009) to circumvent
the intractable normalizing constant problem in random fields. See also Andrieu and Roberts (2009); Maclaurin and Adams (2014) for one of the approximated likelihood in MCMC. The closed form of (16) is given by

\[
P(X^d_i | p_i, \vec{G}_p) = \pi^{\frac{-r_d(i)}{2}} \frac{\mid \Sigma_d_i \mid^{\left(\frac{\nu+1}{2}\right)}}{\mid \Sigma_d \mid^{\frac{\nu}{2}}} \prod_{i=1}^{\mid d_i \mid} \Gamma\left(\frac{\nu+T_i+1-i}{2}\right)
\]

where \(d_i \in \{y_i, \pi_i\}, \pi_i\), and \(X^d_i\) is a sub-matrix of \(X\) consisting of \(|d_i| \times T_i\) realizations, where \(|d_i|\) is the dimension of \(d_i\), \(T_i = T - p_i\), \(\mid \Sigma_d_i \mid\) and \(\mid \Sigma_d \mid\) are the determinants of the prior and posterior covariance matrices associated with \(d_i\).

4.1. Posterior Approximation

Inferring jointly the lag and the graph allows for selecting the relevant predictors to estimate the model parameters \((B, \Sigma_{\varepsilon})\). In order to approximate the posterior distributions of the equations of interest, the standard approach is to consider a collapsed Gibbs sampling. At the \(j\)-th iteration, the sampler consists of the following steps:

1. Sample jointly, \(p^{(j)}, \eta^{(j)}\) and \(\vec{G}_p^{(j)}\) from \(P(p, \eta, \vec{G}_p | X)\).
2. Estimate \(B^{(j)}\) and \(\Sigma^{(j)}_{\varepsilon}\) directly from \(P(B, \Sigma_{\varepsilon} | p^{(j)}, \vec{G}_p^{(j)}, X)\).

As regards to the first step, standard MCMC algorithms (Madigan and York, 1995) such as Gibbs sampling and Metropolis-Hastings (MH) apply only to model selection with fixed dimensions. In model selection problems with unknown lag order, the dimension of the model varies with the lag order. The algorithm extensively applied for this problem is the reversible jump (RJ) MCMC (Green, 1995). In graphical models especially, the space of possible graphs increases super-exponentially with the number of variables (Chickering et al., 2004). Therefore, sampling from a distribution on a union of varying graph dimension using the RJ algorithm will require a higher number of iterations to thoroughly search the space of all possible graphs. In our graphical VAR, the inferential difficulty increases due to the random fan-in restriction.

We propose an alternative algorithm for sampling the graph taking into consideration the random fan-in and estimating the lag order. At the \(j\)-th iteration of the Gibbs, we consider for each equation \(i = 1, \ldots, n_y\) and each lag order \(p_i = p, \ldots, \bar{p}\), a sample of \(\eta_i^{(j)}\) and \(\vec{G}_{p,i}^{(j)}\) from \(P(\eta_i, \vec{G}_{p,i} | p_i, X) \propto P(\eta_i | p_i)P(\pi_i | p_i, \eta_i)P(X | p_i, \vec{G}_{p,i})\). By conditioning on each possible lag order, we are able to apply standard MCMC algorithm thereby avoiding movement between models of different dimensions since the dimension is fixed for each lag. After \(J\) iterations, we average the draws, \(\vec{G}_{p,i}^{(j)}\), over \(J\) and estimate \(\hat{G}_{p,i}\), for each \(p_i = p, \ldots, \bar{p}\), using the criterion discussed in Appendix B.2. This procedure estimates the local graph for the possible lags of \(p_i \in \{p, \ldots, \bar{p}\}\). Next, we find \((\hat{p}_i, \hat{G}_{p,i})\)
minimizes a penalized local log-likelihood (BIC) score given in (22). Given the estimated
graph and lag order, \((\hat{p}_i, \hat{G}_{p,i})\), we select the relevant predictors for each equation to
estimate the model parameters \((B, \Sigma_\epsilon)\).

4.2. Graphical Model Selection

Graphical model selection is a challenge since the dimension of the graph space to
explore increases super-exponentially with the number of variables. In this paper we
apply MCMC and build on the MCMC algorithm described in Grzegorczyk and Husmeier
(2011); Madigan and York (1995). Our algorithm differs from that of the above mentioned
papers in two aspects: the initialization and the inclusion of the random fan-in restriction.

As regards to the initialization, we propose a strategy which improves the mixing of
the chain. In MCMC search algorithms the space exploration crucially depends on the
choice of the starting point of the chain. A set of burn-in chain iterations is often used to
have a good starting point. However, Brooks et al. (2011) pointed out that any sample
that is believed to be representative of the equilibrium distribution is an equally good
starting point. In view of this, we propose an initialization which extracts variables (and
their lags) with reliable information to improve predictions of the dependent variables.
Let \(\vec{G}_{p,i}\) denote the local graph of the \(i\)-th equation, \(V_{p,x}^i\), the vector of all possible
explanatory variables with lags up to \(p\) for each equation, with \(p \in [p, \ldots, \bar{p}]\), and \(V_y\),
the vector of dependent variables. We run the following steps:

1. Initialize the graph \(\vec{G}_{p}\) as \(n_y \times np\) zero matrix, i.e, \(\vec{G}_{p,i}\) is \(1 \times np\) zero vector.
2. For each equation, \(i = 1, \ldots, n_y\):

   2a. Test whether or not predictions of \(y_i \in V_y\) is improved by incorporating in-
       formation from each \(x_k \in V_{p,x}^i\), i.e, \(P(y_i|x_k) > P(y_i)\). Following a Minnesota
type of prior, we assume recent lags (specifically lag 1) of dependent variables
are more reliable to influence current realizations. Based on this idea, we set
\(\vec{G}_{p}(y_i, x_k) = 1\) if \(x_k\) is equal to lag 1 of \(y_i\), and retain \(x_k\) in \(V_{p,x}^i\).

   2b. For \(x_k\) not equal to lag 1 of \(y_i\), we compare the probability of the null hy-
       pothesis, \(H_0 = P(X|p_i, \vec{G}_{p}(y_i, \emptyset))\), where \(\emptyset\) denote the empty set, against the
probability of the alternative, \(H_1 = P(X|p_i, \vec{G}_{p}(y_i, \{x_k\}))\). If \(H_1 > H_0\), we
reject the null, set \(\vec{G}_{p}(y_i, x_k) = 1\) and retain \(x_k\) in \(V_{p,x}^i\). If \(H_1 \leq H_0\), we set
\(\vec{G}_{p}(y_i, x_k) = 0\) and remove \(x_k\) from \(V_{p,x}^i\).

3. We then denote \(N_p(\pi_i)\) as the set of variables, \(x_k's\), retained in \(V_{p,x}^i\).

In our experience, the above initialization provides a good starting point for graphical
model selection. See Figure B.5 for a comparison of the convergence diagnostics of a
random initialization MCMC and our initialization for the graph simulation. Using the
set $N_p(\pi_i)$ of candidate predictors of the dependent variable of the $i$-th equation, we start our MCMC search algorithm. We proceed with the local causal search by investigating the combination of variables in $N_p(\pi_i)$ that produces the highest scoring local graph(s).

As regards the inclusion of the random fan-in restriction, we denote with $m_p = \min\{np,T-p\}$, the maximal number of predictors required to efficiently estimate the model, for $p \in [\bar{p}, \ldots, \bar{p}]$. At the $j$-th iteration, let $\tilde{G}^{(j-1)}_{p,i}$ be the current local graph and $\pi^{(j-1)}_i$, the current set of predictors in $\tilde{G}^{(j-1)}_{p,i}$, then for each equation, $i = 1, \ldots, n_y$, the Gibbs iterates the following steps:

1. Draw the sparsity parameter for the forward proposal probability, $\bar{\eta}^{(s)}_i$ from a $\text{Be}(a,b)$ and set the fan-in $f^{(s)}_i = \lfloor m_p \bar{\eta}^{(s)}_i \rfloor$.

2. If the number of elements in $\pi^{(j-1)}_i$ is less than the fan-in, i.e. $|\pi^{(j-1)}_i| < f^{(s)}_i$, then randomly draw a $x_k$ from the set of candidate predictors, $N_p(\pi_i)$, and add or remove the edge between $y_i$ and $x_k$, i.e. $\tilde{G}^{(s)}_p(y_i,x_k) = 1 - \tilde{G}^{(j-1)}_p(y_i,x_k)$. Here we set the forward proposal probability to $Q(\tilde{G}^{(s)}_p,\tilde{G}^{(j-1)}_{p,i},\bar{\eta}^{(s)}_i) = 1/|N_p(\pi_i)|$. If $|\pi^{(j-1)}_i| \geq f^{(s)}_i$, then randomly draw a variable, $x_k$, from the current set of predictors, $\pi^{(j-1)}_i$, and remove the edge between $y_i$ and $x_k$, i.e. $\tilde{G}^{(s)}_p(y_i,x_k) = 0$.

3. To obtain the reverse proposal probability, we denote $\pi^{(r)}_i$, the set of predictors in $\tilde{G}^{(s)}_p$ taking into consideration the changes made in step 2.

4. Next, we draw the sparsity parameter for the reverse proposal probability, $\bar{\eta}^{(r)}_i$ from a $\text{Be}(a,b)$ and set $f^{(r)}_i = \lfloor m_p \bar{\eta}^{(r)}_i \rfloor$.

5. If $|\pi^{(r)}_i| < f^{(r)}_i$, the reverse move will involve a random draw of a variable from $N_p(\pi_i)$ to add or delete from $\tilde{G}^{(s)}_p$. Thus, the reverse proposal probability is given by $Q(\tilde{G}^{(j-1)}_{p,i})|\tilde{G}^{(s)}_{p,i},\bar{\eta}^{(r)}_i) = 1/|N_p(\pi_i)|$. If $|\pi^{(r)}_i| \geq f^{(r)}_i$, the reverse will randomly draw a variable from $\pi^{(r)}_i$ to delete from $\tilde{G}^{(s)}_p$. The reverse proposal probability in this case is given by $Q(\tilde{G}^{(j-1)}_{p,i})|\tilde{G}^{(s)}_{p,i},\bar{\eta}^{(r)}_i) = 1/|\pi^{(r)}_i|$.

6. From equation (10), the ratio of the local graph priors simplifies to 1 and the acceptance probability is given by $A(\tilde{G}^{(s)}_p,\bar{\eta}^{(s)}_i,\tilde{G}^{(j-1)}_{p,i},\bar{\eta}^{(r)}_i) = \min\{1, R_A\}$ where

$$R_A = \left\{ \frac{P(X|p_i,\tilde{G}^{(s)}_p)}{P(X|p_i,\tilde{G}^{(j-1)}_{p,i})} \frac{Q(\tilde{G}^{(j-1)}_{p,i})|\tilde{G}^{(s)}_{p,i},\bar{\eta}^{(s)}_i)}{Q(\tilde{G}^{(j-1)}_{p,i})|\tilde{G}^{(s)}_{p,i},\bar{\eta}^{(r)}_i)} \right\}$$

(18)

where $P(X|p_i,\tilde{G}^{(s)}_p) = P(X|p_i,\tilde{G}^{(j-1)}_{p,i})$, and can be computed from equations (16) and (17). Note that without the fan-in restriction, the proposal distribution is symmetric, thus, the prior and inverse proposal ratio in (18) simplifies to 1.

7. Sample $u \sim U[0,1]$ and if $u < \min\{1, R_A\}$, then accept changes made in the local graph and set $\tilde{G}^{(j)}_{p,i} = \tilde{G}^{(s)}_p$, otherwise reject changes and set $\tilde{G}^{(j)}_{p,i} = \tilde{G}^{(j-1)}_{p,i}$. 

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A description of the pseudo-code for the graph selection is given in Appendix D.

4.3. Duality between Priors and Penalties

Thanks to the duality between prior distributions and the penalization of likelihood functions, it is possible to define an information criterion for choosing the optimal lag order and estimating the graph for graphical VAR models. This criterion has been used in the first step of our Gibbs sampler (see Section 4.1) and is defined as:

\[
(\hat{p}, \hat{G}) = \arg \max_{p, \bar{G}_p} P(p)P(\bar{G}_p|p)P(X|p, \bar{G}_p).
\]  

(19)

Several authors have considered extensions of the BIC to sparse model selection problems (see Bogdan et al., 2004; Chen and Chen, 2008; Foygel and Drton, 2011). Our extension allows for a more stringent penalty to address the tendency of the BIC to select large size models when dealing with high-dimensional data. To define our lag and graph selection criteria, we proceed by integrating out the hyper-parameter, \(\bar{\eta}_i\), analytically as follows.

**Proposition 2.** For choices of the prior distribution \(P(\bar{\eta}_i|p_i)\) as beta distributed in (8), and \(P(\pi_i|p_i, \bar{\eta}_i)\) according to (10), with \(|\pi_i| = k\), \(\bar{\eta}_i\) can be explicitly marginalized out as

\[
P(\pi_i|p_i) = \frac{1}{2^{n\eta}} \sum_{j=0}^{m_p-1} \sum \left( I_{\{0,...,j\}}(|\pi_i|) \left( I_{\frac{j+1}{m_p}}(a,b) - I_{\frac{j}{m_p}}(a,b) \right) \right)
\]  

(20)

where \(I_z(a,b) = \int_0^z (B(a,b))^{-1}(\bar{\eta})^{a-1}(1 - \bar{\eta})^{b-1} d\bar{\eta}_i\), is the incomplete beta function (Abramowitz and Stegun, 1964, p. 263).

**Proof.** See Appendix A.2.

**Corollary 4.1.** A uniform prior on \(\bar{\eta}_i\), means \(a = b = 1\) and yields

\[
P(\pi_i|p_i) = \frac{1}{2^{n\pi}} \left( 1 - \frac{|\pi_i|}{m_p} \right)
\]  

(21)

**Proof.** See Appendix A.3.

**Proposition 3.** Let \(P(\pi_i|p_i)\) be the local graph prior given in (21) evaluated at the values of \(\pi\) such that \(|\pi_i| = k\). If \(\varphi(k) = -\log P(\pi_i|p_i)\) is considered a function of \(\pi\), with \(|\pi_i| = k\), \(k = 0, \ldots, n_p\), then \(\varphi(k)\) is a convex function given \(p > 0\) and \(n > 0\).

**Proof.** See Appendix A.4.
From (21), it follows that \( P(\pi_i|p_i) \leq \frac{1}{2np}, \forall |\pi_i| \leq m_p \). We define a criteria for graph and lag order selection for each equation the following

\[
BIC(p_i, \hat{G}_{p,i}) = -2 \log P(X|p_i, \hat{G}_{p,i}) + |\pi_i| \log T + 2np_i \log 2
\]  
(22)

where \( \hat{G}_{p,i} \) is the \( i \)-th equation local graph, \( \pi_i \) the set of selected predictors and \( |\pi_i| \) the number of variables in \( \pi_i \). Following a similar approach proposed by Chib and Greenberg (1995), we use the estimated graph \( \hat{G}_{p,i} \) to evaluate the score and to select the lag order \( \hat{p}_i \). Selecting the local graph and lag order for each equation may automatically produce asymmetric lags for the different equations.

4.4. Model Estimation

The posterior of \( \Omega \) conditional on the lag order \( \hat{p} \) and a given graph \( \hat{G}_p \) is Wishart distributed (see Geiger and Heckerman, 2002). Since \( \Omega \) is directly related to \( B \) and \( \Sigma_\epsilon \), (see Proposition 1), we proceed with the posterior estimation of the model parameters focusing on \( B \) and \( \Sigma_\epsilon \). Thus we estimate \( \hat{B} \) and \( \hat{\Sigma}_\epsilon \) from \( P(B, \Sigma_\epsilon|\hat{p}, \hat{G}_p, X) \) with an independent normal-Wishart. By conditioning on \( \hat{G}_p \), we estimate the parameters \( \{\hat{B}_{G,i}, \hat{V}_{G,i}\} \) that corresponds to the non-zero elements of the \( i \)-th equation graph \( \hat{G}_{p,i} \). We define the selection matrix \( E_i = (e_{j_1}, \ldots, e_{j_{|\pi_i|}}) \), where \( E_i \) is of dimension \( np \times |\pi_i| \), \( j_k \) is an element of the set of predictors of the \( i \)-th equation, and \( e_k \) is the standard orthonormal basis of the set of real \( np \)-dimensional vectors. The posterior mean and variance of \( \{\bar{B}_{G,i}, \bar{V}_{G,i}\} \) is given by

\[
\bar{B}_{G,i} = \bar{V}_{G,i}(\bar{V}_{G,i}^{-1}B_{G,i} + \tilde{\sigma}_i^{-2}W_{G,i}Y_i)
\]  
(23)

\[
\bar{V}_{G,i} = (\bar{V}_{G,i}^{-1} + \tilde{\sigma}_i^{-2}W_{G,i}'W_{G,i})^{-1}
\]  
(24)

with

\[
W_{G,i} = WE_i, \quad B_{G,i} = BE_i, \quad V_{G,i} = E_i'Y_iE_i
\]  
(25)

where \( W_{G,i} \in W' \), is the set of selected predictors of the \( i \)-th equation; \( W' \) is stacked \( W'_1, \ldots, W'_{T_0} \), such that \( W' \) is of dimension \( T_0 \times np \); \( Y' \) is stacked \( Y'_1, \ldots, Y'_{T_0} \), such that \( Y' \) is of dimension \( T_0 \times n_y \); \( Y_i \) is the \( i \)-th column of \( Y \); \( B_{G,i} \) and \( V_{G,i} \), are the prior mean and variance of \( W_{G,i} \), respectively; \( \tilde{\sigma}_i^2, i = 1, \ldots, n_y \), is the variance of residuals from the posterior of \( \Sigma_\epsilon \), where the posterior of \( \Sigma_\epsilon^{-1} \) is Wishart distributed with scale matrix

\[
\bar{S} = \bar{S} + (Y' - \bar{B}W')'(Y - W\bar{B})
\]  
(26)

and degrees of freedom \( \bar{\nu} = \nu + T_0 \), where \( \bar{B} = (\bar{B}_{G,1}, \ldots, \bar{B}_{G,n_y}) \), the stacked posterior mean of the coefficients, such that \( \bar{B} \) is of dimension \( n_y \times np \) with positions of non-zero
elements corresponding to non-zero elements in $\hat{G}_p$.

5. Simulation Study

5.1. Metrics for Performance Evaluation

We investigate the effectiveness of our graphical approach with the sparsity prior against one without sparsity prior together with the LASSO of (Tibshirani, 1996) and the Elastic-net (ENET) of Zou and Hastie (2005). We evaluate the efficiency of the algorithms in terms of the estimated graph, the predictive performance of the estimated models on out-of-sample observations and computational cost in terms of run time.

Given the graph of the data generating process (DGP), we extract from the estimated graph the number of true links correctly predicted as $TP$; $FP$ as number of true zero edges predicted as positives; $TN$ as number of true zero edges correctly predicted; and $FN$ as number of true links unidentified. We evaluate the graph estimation performance based on the number of predicted positive links ($PP = TP + FP$), the graph accuracy ($ACC$) and precision ($PRC$) given as

\[
ACC = \frac{TP + TN}{TP + TN + FP + FN} \quad PRC = \frac{TP}{TP + FP} \tag{27}
\]

Furthermore, we evaluate the graph estimation performance in terms of log-likelihood and BIC scores. Following the expression in (22), the graph BIC is obtained as

\[
BIC_G = \sum_{i=1}^{n_y} BIC(p_i, \hat{G}_{p,i}) = -2L_G + \sum_{i=1}^{n_y} (|\hat{\pi}_i| \log T + 2n\hat{p}_i \log 2) \tag{28}
\]

where $L_G = \sum_{i=1}^{n_y} L_i$, with $L_G$ is the log-likelihood of the estimated graph and $L_i = \log P(X|p_i, \hat{G}_{p}(y_i, \pi_i))$ is the log-likelihood of the local graph of the $i$-th equation.

We evaluate the model estimation performance based on the out-of-sample joint density and point forecasts. The log-predictive score (LPS) is the most common measure of the joint predictive density discussed in the literature. Since the competing models might have different number of variables and lags across the equations, the predictive AIC presents a meaningful comparison for purposes of parsimony and is given by

\[
AIC_M = -2 \log P(Y_{\tau_1}|X_{\tau_0}; \hat{B}, \hat{\Sigma}_e) + 2|\hat{B}| \tag{29}
\]

for $\tau_1 = \tau_0 + 1, \ldots, T$, where $\tau_0$ is the number of observations for the training sample, $X_{\tau_0}$ is the training sample dataset; $Y_{\tau_1}$ is the out-of-sample observations of the dependent variables; $|\hat{B}|$ is the number of non-zero coefficients in $\hat{B}$; $\hat{\Sigma}_e$ is the estimated error covariance matrix; and $\log P(Y_{\tau_1}|X_{\tau_0}; \hat{B}, \hat{\Sigma}_e)$ is the log predictive score.
For point forecast, the mean squared forecast error (MSFE) is the most common measure discussed in the literature. To compare the joint point forecasts, we compute the mean MSFE (MMSFE) following

\[
MMSFE = \frac{1}{T - \tau_0 - 1} \sum_{\tau_1 = \tau_0 + 1}^{T} \left( \frac{1}{n_y} \sum_{i=1}^{n_y} (Y_{t_i}^{\tau_1} - \hat{Y}_{t_i}^{\tau_1})^2 \right)
\]

(30)

where \(Y_{t_i}^{\tau_1}\) and \(\hat{Y}_{t_i}^{\tau_1}\) are the out-of-sample observed and predicted values of the \(i\)-th dependent variable respectively.

5.2. Simulation Study Set-up and Results

The set-up of the data generating process (DGP) of the simulated study is as follows

\[
Y_t = BX_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim iid \sim N(0, I_{n_y})
\]

(31)

\(t = 1, \ldots, T\), where \(I_{n_y}\) is \(n_y\) dimensional identity matrix, \(B\) is \(n_y \times n\) coefficient matrix, \(Y_t\) and \(X_t\) are a \(n_y \times 1\) and \(n \times 1\) respectively. To analyze different sparsity levels, we generate the coefficients matrix \(B\) such that, the number of non-zero coefficients for each equation is drawn from a uniform on \(\{0, \ldots, 40\}\). We considered a large dimensional model by setting \(n_y = 10, n = 100\). We replicate the simulation and estimation exercises 100 times. The 100 replications have been conducted on a cluster multiprocessor system which consists of 4 nodes; each comprises four Xeon E5-4610 v2 2.3GHz CPUs, with 8 cores, 256GB ECC PC3-12800R RAM, Ethernet 10Gbit, 20TB hard disk system with Linux. The simulation study in Table 1 takes about 14 minutes of CPU time. For each replication, we generate a sample size, \(T = 60\) and use \(T_0 = 50\) for model estimation and 10 for out-sample forecast analysis.

We run 20,000 Gibbs iterations for the graph estimation and 2000 iterations for parameter estimations. We applied the standard approach of (Tibshirani, 1996) and Zou and Hastie (2005) for the LASSO and ENET estimation respectively. We set \(p = 1\) and \(\bar{p} = 4\) and implement a parallel estimation for the LASSO and ENET. For each \(p \in [\underline{p}, \bar{p}]\), we sequentially use one variable as the dependent variable and the remaining as the predictors. We apply a five-fold cross validation to select the regularization parameter \(\lambda\) with minimal plus one standard error point (index1SE).

Figure B.6 shows the convergence diagnostics of the graph simulation and the local graph BIC for the lags. The figure of the PSRF indicates convergence of the chain. We also notice from Figure B.6d that the posterior distribution on the lag order for each equation of the simulation experiment using our modified BIC favors lag order \(p = 1\).

We report in Table 1, the performance of the LASSO, ENET, BGVAR and SBGVAR for
the inference of the DGP in (31).

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<th>LASSO</th>
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<th>SBGVAR</th>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>PP</td>
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<td>131.30</td>
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<td>51.85</td>
<td>39.31</td>
<td><strong>69.58</strong></td>
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<td><strong>1307.80</strong></td>
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<td>-236.79</td>
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<td>972.42</td>
<td><strong>472.74</strong></td>
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<tr>
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<td>0.69</td>
<td>0.62</td>
<td><strong>0.59</strong></td>
</tr>
<tr>
<td>Time (in seconds)</td>
<td>55.76</td>
<td>50.18</td>
<td>162.52</td>
<td><strong>42.47</strong></td>
</tr>
</tbody>
</table>

Table 1: Average graph and model estimation performance of algorithms over 100 replications. PP - number of predicted positive links; TP - number of true positive links; ACC - graph accuracy; PRC - graph precision; $L_G$ - graph log-likelihood; BIC$_G$ - graph BIC; LPS - log predictive score; AIC$_M$ - predictive AIC; and MMSFE - mean of MSFE. Bold values indicate the best choice for each metric.

We proceed by comparing the effectiveness of the algorithms in estimating the graph of the true DGP, when the DGP average links number is 201.5. Table 1 shows that without the sparsity prior distribution, the BGVAR overestimates the number of links compared to the other algorithms. The Lasso-type methods (LASSO and Elastic-Net) fall in the middle with a lower number of links compared to that of the DGP. The SBGVAR on the other hand recorded the least number of edges. This is quite expected since the idea is to select the subset of the explanatory variables that explains a large variation in the dependent variables.

By including more edges than the true DGP, the graphical search algorithm without sparsity prior (BGVAR) records the highest true positive links but relatively low accuracy and precision compared to the other algorithms. Again the Lasso-type methods fall in the middle, recording a lower number of true positive links but with a higher accuracy and precision than the BGVAR. The sparsity prior graphical approach instead had the least number of true positive edges but tends to be more accurate and much precise than the other algorithms. The log-likelihood score of the graph favored the BGVAR but the graph BIC score favored the SBGVAR. Thus the BIC score confirms the outcome of the graph accuracy metric which shows that though the SBGVAR records the least edges, it produced a better representation of the temporal dependence in the simulated dataset than the Lasso-type methods and the BGVAR.

The log predictive score, predictive AIC and the MMSFE all favor the SBGVAR over the other competitors. One would expect the Lasso-type methods to perform better than the graphical VAR, however this is not the case according to the above simulation results. This is attributable to the fact that the Lasso-type techniques perform both
model selection and parameter estimation simultaneously. This may seem to be an advantage but on the other hand it affects the estimated parameters, since it shrinks all coefficients at the same rate (see Gefang, 2014). In addition, the Lasso-type methods only focus on estimating the coefficients in each equation neglecting the interaction among the errors across the different equations. The graphical approach instead focus on selecting and estimating only the coefficients of relevant variables taking into consideration the interaction among the errors across the different equations. Thus the latter achieves better parameter estimation efficiency than the Lasso-type models. The result shows that the sparsity prior on the graph enables us to identify the small set of the most influential explanatory variables that explains a large variation in the dependent variables. Also, the SBGVAR produce a more parsimonious model with better out-of-sample forecasts than the Lasso-type methods.

On the computational intensity, the SBGVAR spends less time than the other algorithms. Interestingly, it records about one-fourth of the run time of the BGVAR. Thus, the sparsity prior of the fan-in restriction helps to reduce the run time by considering a relatively lower search space in terms of the number of combinations of explanatory variables. The higher run time of the Lasso-type methods is due to the cost of cross-validation to select the regularization parameter.

5.3. Sparsity and Indeterminacy Evaluation

A system of linear equations is said to be under-determined (or indeterminate) when the number of parameters to estimate exceeds the number of observations (see Donoho, 2006). Such systems can be modeled by exploiting sparsity. Here, we investigate the performance of the graphical model approaches against the standard Lasso-type methods for different level of indeterminacy and sparsity of the DGP.

For a VAR model with \( n_y \) dependent variables and \( n \) explanatory variables for each equation, with a maximum lag order \( p \), we have a total of \( n_y np \) number of coefficients to estimate. Given a multiple time series with \( T \) observations, the total number of observations of the dependent variables is given by \((T - p)n_y\). Following Donoho (2006), we measure the level of indeterminacy by \( \delta = (T - p)n_y/np = (T - p)/np \), and the level of sparsity by \( \rho = k n_y/(T - p)n_y = k/(T - p) \), where \( k \) is the number of non-zero coefficients in each equation of the DGP.

Following Donoho and Stodden (2006), we formulate our experiment by setting the DGP to generate a VAR model with \( n_y = 10 \), \( n = 100 \) and lag order \( p = 1 \). For different level of indeterminacy, we set \( T - p \) to take values \( \{20, \ldots, 100\} \). For each \( T - p \), we generate for each equation, \( k = \lceil \rho(T - p) \rceil \), where \( \rho \) takes values \( \{0.2, 0.3, \ldots, 1\} \). This is to allow for different sparsity levels for each level of indeterminacy.
We proceed by comparing the effectiveness of the LASSO, ENET, BGVAR and SBGVAR in estimating the true DGP by setting $p = \bar{p} = 1$. For each $T$ and $k$, we replicate the simulation and estimation exercise 10 times with the magnitude of the coefficients drawn from a uniform on $[-1, 1]$. In each replication, we estimate the model and perform a 1-step ahead forecast. Figure 1 shows the estimation performance of the algorithms for the different levels of indeterminacy averaged over the different levels of sparsity.

Figure 1: Estimation performance of the algorithms for different levels of indeterminacy averaged over different levels of sparsity. The LASSO is in green, ENET in blue, BGVAR in red and SBGVAR in cyan.

Figure 1a shows the difference between the average DGP number of links and the
estimated links of the algorithms. Except for the BGVAR, all the other algorithms estimated a lower number of links compared to that of the DGP. More specifically, the BGVAR seems to overestimate the number of DGP links for lower under-determined models, whereas the SBGVAR underestimates the number of DGP links regardless of the level of indeterminacy. We also see that, the difference between the DGP and the estimated links of the BGVAR and SBGVAR increases overtime while the Lasso-type methods are relatively stable regardless of the level of indeterminacy.

The graph accuracy in Figure 1b shows that all the algorithms experienced a deterioration in the accuracy of the prediction of the graph associated with the DGP. However, on average the SGBVAR performs slightly better at the graph estimation for lower under-determined models than the Lasso-type methods.

In Figure 1c, the graph BIC of the algorithms increases with the level of indeterminacy. This is not surprising since the BIC is a direct function of the number of observations which increases with the level of indeterminacy. Again we observe that the graph BIC score favors the graph estimated by the SBGVAR over the other competing algorithms. This shows that though the SBGVAR recorded the minimum number of links, it produce a better representation of the graph associated with the DGP.

For model estimation performance, Figure 1d shows that all the algorithms perform better at out-of-sample point forecasts for higher under-determined models. The MMSFE of the algorithms are not significantly different though we find that it favors the SBGVAR for lower under-determined models. The predictive AIC (in Figure 1e) on the other strongly favors the SBGVAR for all level of indeterminacy.

On the computational intensity, we notice (from Figure 1f) an increase in run time with the level of indeterminacy for all algorithms except the BGVAR which seems slightly constant over time. Overall, the Lasso-type methods achieve a lower run time for lower under-determined models whiles the SBGVAR achieves lower run time for higher under-determined models.

We focus attention on the model estimation performance of the algorithms for the different levels of indeterminacy and sparsity. Figure 2 shows the heatmap of the predictive AIC of the models of the algorithms estimated over the levels of sparsity and indeterminacy of the DGP. The color bar shows the different range of values of the predictive AIC, where blue represents lower AIC, and red for highest AIC. Clearly, we notice a significant difference between the results of the Lasso-type methods and that of the graphical model approaches. Thus the LASSO and ENET are not significantly different from each other, whiles the BGVAR and SBGVAR are quite different, dominated by cyan and blue respectively. The figure shows that the predictive AIC favor the SBGVAR over all levels of sparsity and indeterminacy. The Lasso-type methods only performs better than the
BGVAR for lower under-determined models with different level of sparsity, while the BGVAR dominates in higher under-determined models.

The results of this exercise confirm that of our first simulation experiment. Firstly, the sparsity prior on the graph space induces sparsity on the estimated graph of the temporal relationship among the variables. Secondly, the random fan-in restriction helps to reduce the computational complexity by considering a relatively lower search space in terms of the number of combinations of explanatory variables. Thirdly, though the SBGVAR under-estimates the number of links compared to the DGP and other algorithms, it is able to identify the small set of the most influential explanatory variables that explains a large variation in the dependent variables. Thus, the SBGVAR produces a more parsimonious
model with competitive out-of-sample joint point forecasts and better density forecasts than the competing models.

6. Forecasting VAR with Many Predictors

Several studies have shown empirically that applying large VAR models for macroeconomic time series produces better forecasts than standard approaches (see Banbura et al., 2010; Carriero et al., 2013; Giannone et al., 2005; Koop, 2013; Stock and Watson, 2012). In the literature, researchers typically work with a single model with fixed or time varying coefficients (see Koop and Korobilis, 2013). It is therefore important to allow for changes in structure and/or parameters to understand the dynamic evolution of the relationship among variables. As part of our contribution, we apply our graphical scheme to model and forecast selected macroeconomic variables with large number of predictors.

The dataset is quarterly observations of 130 US-macroeconomic variables. All series were downloaded from St. Louis’ FRED database and cover the quarters from 1959Q1 to 2014Q3. Some series had missing observations which are completed with earlier version of the database used by Korobilis (2013). We follow the adjustment codes of De Mol et al. (2008); Stock and Watson (2012) and Korobilis (2013) to transform all the series into stationarity. See Appendix C for the list of series and adjustment codes. We consider 6 series as dependent variables and the remaining 124 as predictors. The dependent variables are: consumer price index (CPIAUCSL), Federal funds rate (FEDFUNDS), real gross domestic product (GDPC96), real gross private domestic investment (GDPIC96), industrial production index (INDPRO) and real personal consumption expenditure (PCEC96).

We set the minimum and maximum lag order equal to $p = 1$ and $\bar{p} = 4$ respectively according to the literature. We consider a moving window with a starting sample from 1960Q1 to 1970Q4 to estimate the model and to forecast 1 to 4-quarters ahead. We then move the window forward by 4-quarters. Our last sample covers 2003Q1 to 2013Q4, and the final forecast is up to 2014Q3.

Figures B.7 in Appendix B show the convergence diagnostics of the graph simulation and the local graph BIC for the lags for the macroeconomic application. The figure of the PSRF indicates convergence of the chain. Clearly, the global log score of the graph seems to increase with the lag order in Figure B.7b whereas the total number of links of the different lags seems quite close as displayed in Figure B.7a. However, we notice from Figure B.7d that the posterior distribution on the lag order for each equation of the macroeconomic application using our modified BIC score favors lag order $p = 1$.

We report in Figure 3, the graph and model estimation performance of the Lasso-type methods and the graphical VAR approaches in modeling and forecasting the selected
Figure 3: Performance of the algorithms in modeling and forecasting selected macroeconomic variables with many predictors over the sample period 1960Q1 – 2014Q3. Figures 3a - 3c show the graph estimation performance, whilst 3d - 3f depict the model estimation performance.

macroeconomic variables over the sample period 1960Q1 – 2014Q3. The graph estimation performance is compared in terms of the number of link (PP - predicted positive edges), the log-likelihood of the graph (LG) and the BIC score of the graph (BICG). The model estimation performance is compared in terms of the log predictive score (LPS), the predictive AIC (AICM) and the average of the mean squared forecast errors (MMSFE).

Table 2 presents the averages of the graph and model estimation performance of the
algorithms including the computational time over the sample period.

From Figure 3, we observe that the BGVAR estimated more edges than the other algorithms in a greater part of the sample period. This is followed by the Lasso-type methods, (ENET, then the LASSO) and the SBGVAR records the least number of links over the entire sample period. In scoring the estimated graphs, the BGVAR obtained the highest log likelihood over the entire period whiles the SBGVAR records the minimum at the beginning but showed significant improvement over the rest of the sample period. The BIC score of the graph however favored the SBGVAR over the other algorithms. The summary of the averages in Table 2 shows that the by including more edges than the other algorithms the BGVAR records the highest log likelihood of the graph whilst the SBGVAR with the least number of links obtained the minimum BIC score indicating that the SGBVAR graph presents a better representation of the temporal dependence in the macroeconomic application than the Lasso-type methods and the BGVAR.

Figure 3d shows the evolution of the out-of-sample joint point forecasts of the models estimated by the algorithms. We observe from the plot that the MSFE are not very different from each other. However, in terms of the out-of-sample joint density forecasts, the BGVAR model presents the minimum cumulative log predictive score and that of the SBGVAR model dominates the LASSO but is very competitive against the ENET model. When adjusted for the number of selected variables used for the forecasting analysis, the SBGVAR model obtain the minimum predictive AIC whilst the BGVAR model performed worst than the Lasso-type models. From Table 2, we see that on average the Lasso-type models obtain the minimum MMSFE and this indicate that they produce slightly better point forecasts than the graphical approaches. The average log predictive score and AIC on the other hand are in favor of the SGBVAR over the Lasso-type models.

<table>
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<tr>
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<th>ENET</th>
<th>BGVAR</th>
<th>SBGVAR</th>
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<tr>
<td>PP</td>
<td>25.14</td>
<td>39.09</td>
<td>70.82</td>
<td>13.50</td>
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<tr>
<td>LG</td>
<td>372.26</td>
<td>379.45</td>
<td>437.21</td>
<td>400.05</td>
</tr>
<tr>
<td>BICG</td>
<td>434.09</td>
<td>473.71</td>
<td>481.01</td>
<td>333.46</td>
</tr>
<tr>
<td>LPS</td>
<td>-36.87</td>
<td>-34.33</td>
<td>-44.36</td>
<td>-33.79</td>
</tr>
<tr>
<td>AICM</td>
<td>124.01</td>
<td>146.84</td>
<td>230.36</td>
<td>94.57</td>
</tr>
<tr>
<td>MMSFE</td>
<td>1.30</td>
<td>1.26</td>
<td>1.24</td>
<td>1.32</td>
</tr>
<tr>
<td>Time (in seconds)</td>
<td>57.93</td>
<td>42.26</td>
<td>65.74</td>
<td>23.77</td>
</tr>
</tbody>
</table>

Table 2: Average graph and model estimation performance of algorithms in modeling and forecasting selected macroeconomic series from 1960Q1 – 2014Q3. PP - number of predicted positive edges; LG - graph log-likelihood; BICG - graph BIC; LPS - log predictive score; AICM - predictive AIC; and MMSFE - mean of MSFE. Bold values indicate the best choice for each metric.

On the computational intensity, the result shows that on average the SBGVAR spends less simulation time on graph sampling and parameter estimation than the other algorithms. Interestingly, it records about one-fourth of the run time of the BGVAR.
Figure 4: Frequency of inclusion of the most influential variables that explain a large variation in the dependent variables of the macroeconomic application averaged over the sample period 1960Q1—2014Q3. CPIAUCSL is consumer price index, FEDFUNDS - Federal funds rate, GDPC96 - real gross domestic product, GPDIC96 - real gross private domestic investment, INDPRO - industrial production index and PCECC96 - real personal consumption expenditure.
In summary, we find evidence that the graphical VAR approach with our new graph prior distribution induces sparsity on the graph structure. In modeling and forecasting our selected macroeconomic series, the result shows that the SBGVAR better represents the temporal dependence, since it is more parsimonious than the competitors. Furthermore, we find evidence of a gain in the predictive performance of the SBGVAR approach over the Lasso-type methods. It is also less computationally intensive compared to the graphical approach without sparsity prior and the Lasso-type methods.

In Figure 4, we report the frequency of the most influential variables that explain a large variation in the dependent variables of the macroeconomic application, averaged over the sample period 1960Q1 – 2014Q3. For convenience and clarity of presentation, we report only the top explanatory variables of real investment (GPDIC96) with frequency up to 14%. We find strong evidence supporting the effect of financial variables on the real sector of the US economy. More specifically, we find that over the sample period, S&P 500 and exchange rates especially with Japan, Switzerland and UK, have strong effects on real investment (GPDIC96) and industrial production index (INDPRO), and weak effects on real gross domestic product (GDPC96) and on the Federal funds rate (FEDFUNDS). The results are in line with the recommendation by Diebold and Yilmaz (2015) suggesting the importance of monitoring the connectedness between real activity and stock returns (or financial variables). Furthermore, it offers some insight for further evaluation of macro-financial linkages which have long been at the core of the IMF’s mandate to oversee the stability of the global financial system. The figure also shows that apart from real investment and industrial production index, that report a higher number of predictors over the sample period, the rest can be predicted by a handful of macroeconomic variables.

7. Conclusion

This paper develops a Bayesian approach to model dependence in high-dimensional multivariate time series and to address over-parametrization in large vector autoregressive (VAR) models. The methodology discussed in the paper is based on combining graphical model notion of causality with a new sparsity prior distribution on the graph space to address model selection problems in multivariate time series of large dimension. In particular, this work builds on the application of restriction on the explanatory variables (fan-in) in the VAR model by allowing for different prior information level about the maximal number of predictors for each equation. This prior distribution proves to be efficient in reducing the number of possible combinations of predictors to explore for each equation when determining the dependence in a large VAR model. Furthermore,
the Bayesian paradigm allows us to take into account uncertainties about the maximum lag order, the dependence structure and coefficients of the VAR through model averaging.

In both simulation study and empirical macroeconomic application to real datasets, we find evidence that our sparsity prior distribution enables us to control the fraction of explanatory variables with temporal causal effect on the dependent variables. The comparison with the standard Lasso-type methods (LASSO and Elastic-Net) as a benchmark shows that our model is more sparse and parsimonious than the benchmark. The results show that, compared to the competing methods, the sparse graphical approach is able to recover the small set of predictors that explains a large variation in the dependent variables of the large VAR model. More specifically, the BIC score of the graph of temporal dependence and the predictive AIC of the estimated model all favor the sparse graphical VAR model over the Lasso-type methods. Furthermore, we find evidence of a gain in sampling the graph of the temporal dependence among variables which allows to impose zero restrictions supported by the data on the non relevant components of the predictors in order to estimate the coefficients of the selected variables. On the macroeconomic application, we find evidence supporting the effect of financial variables on the real sector of the US economy. Thus, our methodology and result offers insight for further research into empirical evaluation of macro-financial linkages which has long been the core of the IMF’s mandate to oversee the stability of the global financial system.

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Social Systems Research Institute, University of Wisconsin-Madison.
Appendix A. Proofs

Appendix A.1. Proof of Proposition 1

Proof. Let $X_t$ be a $n \times 1$ vector of observations at time $t$, $Y_t \subseteq X_t$ a $n_y \times 1$ vector of dependent variables, $W_t$ the stacked lags of $X_t$, where $W_t = (X_{t-1}', \ldots, X_{t-p}')'$ is of $np \times 1$ dimension, with $p$ as the maximum lag order. Suppose the joint distribution of $(Y_t', W_t') \sim N(\mu, \Omega^{-1})$, where $\mu$ is the $((np + n_y) \times 1)$ vector of means and $\Omega^{-1}$ is $(np + n_y) \times (np + n_y)$ matrix of covariances. Without loss of generalization, we assume $\mu$ is a zero vector.

Suppose the marginal distribution of $W_t \sim N(0, \Sigma_{ww})$ and the conditional distribution of $Y_t | W_t \sim N(BW_t, \Sigma_\varepsilon)$, where $B$ is $n_y \times np$ matrix of coefficients and $\Sigma_\varepsilon$ is $n_y \times n_y$ covariance matrix of the errors. Then given $\Omega$ as the precision matrix of $(Y_t, W_t)$, we can obtain $\Sigma = \Omega^{-1}$ which can be expressed as

$$
\Sigma = \begin{pmatrix}
\Sigma_{yy} & \Sigma_{yw} \\
\Sigma_{wy} & \Sigma_{ww}
\end{pmatrix}
$$

(A.1)

where $\Sigma_{wy}$ is $np \times n_y$ the covariances between $W_t$ and $Y_t$, and $\Sigma_{yy}$ is $n_y \times n_y$ covariances among $Y_t$. Then $B$ and $\Sigma_\varepsilon$ can be obtained from $\Sigma$ by

$$
B = \Sigma_{yw} \Sigma_{\varepsilon}^{-1}, \quad \Sigma_\varepsilon = \Sigma_{yy} - \Sigma_{yw} \Sigma_{\varepsilon}^{-1} \Sigma_{wy}
$$

(A.2)

Now given $\Sigma$ as in (A.1), $\Omega = \Sigma^{-1}$ can be obtained as:

$$
\Omega = \begin{pmatrix}
(\Sigma_{yy} - \Sigma_{yw} \Sigma_{ww}^{-1} \Sigma_{wy})^{-1} & -(\Sigma_{yy} - \Sigma_{yw} \Sigma_{ww}^{-1} \Sigma_{wy})^{-1} \Sigma_{yw} \Sigma_{\varepsilon}^{-1} \\
-(\Sigma_{ww} - \Sigma_{yw} \Sigma_{yy}^{-1} \Sigma_{wy})^{-1} \Sigma_{wy} \Sigma_{yy}^{-1} & (\Sigma_{ww} - \Sigma_{yw} \Sigma_{yy}^{-1} \Sigma_{wy})^{-1}
\end{pmatrix}
$$

(A.3)

To complete the proof, we report the well-known Sherman-Morrison-Woodbury formula (see Woodbury, 1950). Thus, the inverse of a partitioned symmetric matrix is given by

$$
(A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} = A_{11}^{-1} + A_{11}^{-1} A_{12} (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} A_{21} A_{11}^{-1}
$$

$$
-(A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} A_{12} A_{22}^{-1} = -A_{11}^{-1} A_{12} (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}
$$

(A.4)

Following (A.4) and the expressions in (A.2), (A.3) can be simplified as

$$
\Omega = \begin{pmatrix}
\Sigma_{\varepsilon}^{-1} & -\Sigma_{\varepsilon}^{-1} B \\
B^T \Sigma_{\varepsilon}^{-1} & \Sigma_{ww}^{-1} + B^T \Sigma_{\varepsilon}^{-1} B
\end{pmatrix}
$$

(A.5)
Appendix A.2. Proof of Proposition 2

Proof. From the prior distributions in (8) and (10), \( \bar{\eta} \) can be marginalized out as

\[
P(\pi_i | p_i) = \frac{1}{2np_i} \int_0^1 \mathbb{I}_{[0, \ldots, f_i]}(|\pi_i|) \frac{1}{B(a, b)} (\bar{\eta})^{a-1} (1 - \bar{\eta})^{b-1} \, d\bar{\eta}
\]

where \( f_i = \lceil \bar{\eta} m_p \rceil \) with \( m_p = \min \{ np_i, T - p_i \} \), \( \mathbb{I}_{[0, \ldots, f_i]}(|\pi_i|) \) is the indicator function

\[
\mathbb{I}_{[0, \ldots, f_i]}(|\pi_i|) = \begin{cases} 
\mathbb{I}_0(|\pi_i|), & 0 \leq \bar{\eta} < \frac{1}{m_p} \\
\vdots & \\
\mathbb{I}_{0, \ldots, m_p-1}(|\pi_i|), & \frac{m_p-1}{m_p} \leq \bar{\eta} < 1 \\
\mathbb{I}_{0, \ldots, m_p}(|\pi_i|), & \bar{\eta} = 1
\end{cases}
\]

(A.6)

Let \( f(\bar{\eta}) = (B(a, b))^{-1} \bar{\eta}^{a-1} (1 - \bar{\eta})^{b-1} \). From (A.6)

\[
P(\pi_i | p_i) = \frac{1}{2np_i} \left[ \mathbb{I}_0(|\pi_i|) \int_0^{\frac{1}{m_p}} f(\bar{\eta}) \, d\bar{\eta} + \ldots + \mathbb{I}_{0, \ldots, m_p-1}(|\pi_i|) \int_{\frac{m_p-1}{m_p}}^{1} f(\bar{\eta}) \, d\bar{\eta} \right]
\]

(A.8)

where \( I_z(a, b) = \int_0^z f(\bar{\eta}) \, d\bar{\eta} \) is the incomplete beta function (Abramowitz and Stegun, 1964, p. 263).

Appendix A.3. Proof of Corollary 4.1

Proof. By assuming a uniform prior on \( \bar{\eta} \), \( f(\bar{\eta}) = 1 \). Furthermore, the difference between the incomplete beta functions in (A.8) is \( I_{\frac{m_p}{m_p}}(a, b) - I_{\frac{1}{m_p}}(a, b) = \frac{1}{m_p} \). Thus

\[
P(\pi_i | p_i) = \frac{1}{2np_i} \frac{m_p-1}{m_p} \sum_{j=0}^{m_p-1} \mathbb{I}_{0, \ldots, j}(|\pi_i|) = \frac{1}{2np_i} \left( 1 - \frac{|\pi_i|}{m_p} \right)
\]

(A.9)

Appendix A.4. Proof of Proposition 3

Proof. The function \( \varphi(k) \) is convex if and only if \( \varphi''(k) > 0 \), \( \forall k \). By defining \( \varphi(k) = -\log P(\pi_i | p_i) = np_i \log(2) + \log(m_p) - \log(m_p - k) \), for \( |\pi_i| = k \), it can be shown that

\[
\varphi''(k) = \frac{1}{(m_p - k)^2} > 0
\]

(A.10)
Appendix B. Convergence Diagnostics and Posterior Approximation

Appendix B.1. Convergence Diagnostics

For our graphical approach, we monitor the convergence of the MCMC chain using the potential scale reduction factor (PSRF), see Gelman and Rubin (1992). See also Casella and Robert (2004), ch. 12, for a review on methods for convergence monitoring in MCMC. The PSRF monitors the within-chain and between-chain covariances of the global log posterior densities of the sampled structures to test whether the chain is said to have properly converged if $PSRF \leq 1.2$. Figure B.5 display a comparison of the MCMC convergence diagnostics for a random initialization and our initialization procedure of the graph averaged over lags. Figures B.6 and B.7 shows plots of links and graph score at each MCMC iteration, the local graph BIC for the lags for the simulation experiments and the macroeconomic application respectively.

Figure B.5: Comparison of the MCMC convergence diagnostics for a random initialization (in blue) and our initialization (in green) procedure of the graph averaged over lags. The black dashed line is 1.2, and colored lines close to this line indicate convergence of the chain.

Appendix B.2. Edge Posterior Approximation

We estimate the posterior probability of the edge by $\hat{e}_{ij}$, which is the average of the MCMC samples from the $G_{ij}$ posterior distribution. For variable selection purposes, we define the estimator $G^*_{ij}$ of the edge from $X^j$ to $X^i$ based on a one sided posterior credibility interval for the edge posterior distribution, and find the interval lower bound $G^*_{ij} = 1$ if $\hat{e}_{ij} - z_{(1-\alpha)} \sqrt{\frac{\hat{e}_{ij}(1-\hat{e}_{ij})}{n_{eff}}} > 0.5$, where $n_{eff}$ is the effective sample size representing the number of independent posterior samples of the graph, and $z_{(1-\alpha)}$ is the z-score of the normal distribution at the $(1-\alpha)$ significance level.
Figure B.6: Plots of (B.6a) links and (B.6b) graph score at each MCMC iteration, with (B.6c) convergence diagnostics and (B.6d) local graph BIC for the lags for each equation of the simulation experiments.

Figure B.7: Plots of (B.7a) links and (B.7b) graph score at each MCMC iteration, with (B.7c) convergence diagnostics and (B.7d) local graph BIC for the lags for each equation of the macroeconomic application.
Appendix C. Data Description For Large Macroeconomic Application

Table C.3 provides a description and stationarity transformation codes used for our large macroeconomic application in Section 6.

Appendix D. Pseudo-Code for Sparse Graph Selection

Algorithm 1 presents a description of the pseudo-code for the sparse graphical model selection.
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<th>Mnemonic</th>
<th>Description</th>
<th>Code</th>
</tr>
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<tbody>
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<td>1</td>
<td>CPIAUCSL*</td>
<td>Consumer Price Index for All Urban Consumers: All Items</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>FEDFUNDS*</td>
<td>Effective Federal Funds Rate</td>
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Table C.3: Data description and transformation codes. 1 = no transformation, 2 = first difference, 4 = log, 5 = 100*(first difference of log), 6 = 100*(second difference of log). * The dependent variables.
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Algorithm 1 Graphical VAR Model Selection Algorithm

for $p \in [p, \ldots, \bar{p}]$ do
  Initialize $\tilde{G}_p$ as $(n_y \times np)$ zero matrix, set $m_p = \min \{np, T - p\}$
  $V_{p,x}$ the vector of all possible explanatory variables up to lag $p$
  $V_y$ the vector of dependent variables
  for each $y_i \in V_y$ do
    for each $x_k \in V_{p,x}$ do
      if $x_k$ is equal to lag 1 of $y_i$ then
        Set $\tilde{G}_p(y_i, x_k) = 1$
      else
        Compute $H_0 = P(X[p_i, \tilde{G}_p(y_i, \emptyset)])$ and $H_1 = P(X[p_i, \tilde{G}_p(y_i, \{x_k\}])$
        if $H_1 > H_0$ then
          Set $\tilde{G}_p(y_i, x_k) = 1$ and retain $x_k$ in $V_{p,x}$
        else
          Set $\tilde{G}_p(y_i, x_k) = 0$ and remove $x_k$ from $V_{p,x}$
  Set $N_p(\pi_s)$ the set of variables, $x_k$’s, retained in $V_{p,x}$
  for $j = 1 \rightarrow J$, the total iterations do
    for each $y_i \in V_y$ do
      Set $\pi_i^{(j-1)}$ = the set explanatory variables of $y_i$ in $\tilde{G}_p^{(j-1)}$
      Draw $\tilde{\eta}_i^{(s)}$ from a $\text{Be}(a, b)$ and set $f_i^{(s)} = \lfloor m_p \tilde{\eta}_i^{(s)} \rfloor$
      if $|\pi_i^{(s)}| < f_i^{(s)}$ then
        Set $Q(\tilde{G}_p^{(s)}(\pi_i^{(j-1)}, \tilde{\eta}_i^{(s)})) = 1/|N_p(\pi_s)|$
        Draw $x_k \in N_p(\pi_i)$
        Add/remove edge; i.e. $\tilde{G}_p^{(s)}(y_i, x_k) = 1 - \tilde{G}_p^{(j-1)}(y_i, x_k)$
      else
        Set $Q(\tilde{G}_p^{(s)}(\pi_i^{(j-1)}, \tilde{\eta}_i^{(s)})) = 1/|\pi_i^{(j-1)}|$
        Draw $x_k \in \pi_i^{(j-1)}$
        Remove edge; i.e. $\tilde{G}_p^{(s)}(y_i, x_k) = 0$
      Set $\pi_i^{(*)} = \pi_i^{(s)}$ the set explanatory variables of $y_i$ in $\tilde{G}_p^{(*)}$
      Draw $\tilde{\eta}_i^{(*s)}$ from a $\text{Be}(a, b)$ and set $f_i^{(*s)} = \lfloor m_p \tilde{\eta}_i^{(*s)} \rfloor$
      if $|\pi_i^{(*)}| < f_i^{(*s)}$ then
        Set $Q(\tilde{G}_p^{(*s)}(\pi_i^{(j-1)}, \tilde{\eta}_i^{(*s)})) = 1/|N_p(\pi_s)|$
      else
        Set $Q(\tilde{G}_p^{(*s)}(\pi_i^{(j-1)}, \tilde{\eta}_i^{(*s)})) = 1/|\pi_i^{(*)}|$
    Sample $u \sim U_{[0,1]}$
    Compute $R_A$ following equation (18)
    if $u < \text{min}\{1, R_A\}$ then
      $\tilde{G}_p^{(j)} = \tilde{G}_p^{(*)}$
    else
      $\tilde{G}_p^{(j)} = \tilde{G}_p^{(j-1)}$