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Multi-jumps*

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Abstract

We provide clear-cut evidence for economically and statistically significant multivariate jumps (*multi-jumps*) occurring simultaneously in stock prices by using a novel nonparametric test based on smoothed estimators of integrated variances. Detecting multi-jumps in a panel of liquid stocks is more statistically powerful and economically informative than the detection of univariate jumps in the market index. On the contrary of index jumps, multi-jumps can indeed be associated with sudden and large increases of the variance risk-premium, and possess a statistically significant forecasting power for future volatility and correlations which implies a sizable deterioration in the diversification potential of asset allocation.

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1 Introduction

Figure 1 shows the intraday log-returns of four financial stocks (see Table 6) on December, 11th 2007. In that day, a FOMC meeting was taking place, ending with the decision of lowering the target for federal funds rate of 25 basis points, due to "slowing economic growth reflecting the intensification of the housing correction" and "financial strains"¹. The four financial companies collapsed all together in the afternoon, with a contemporaneous log-return of approximately -3% which is clearly visible in the figure. The figure also shows an evident increase, after the collapse, of both the stocks' volatility and their correlation. Moreover, the VIX index rose that day to 23.59 from 20.74 (+13.7%).

In the continuous time literature, a price movement of 3% (when the local volatility is less than 0.5% , thus of more than six standard deviations in volatility units) is typically modeled as a *jump*, that is a discontinuous variation of the price process. There are three possible routes to the detection of collective events like that in Figure 1 in the data: i) detection of a jump in a portfolio which includes the stocks (e.g., the equity index); ii) detection of jumps in individual stocks; iii) direct detection of the multivariate jump (or *multi-jump* as we call it in this paper). Surprisingly, a lot of effort has been devoted to *i)* and *ii)*, both theoretically and empirically, but almost none to *iii)*. In this paper, we introduce a formal test for the detection of multi-jumps, we argue that the third option is actually the most effective and we show that it reveals additional economic information which could not be revealed by the first two.

Multi-jumps are crucial events for asset allocation and risk management, as recognized by the financial literature. For example, Longin and Solnik (2001) show that correlations increase after a collective crash in the market, dampening the diversification potential of portfolio managers, and Das and Uppal (2004) use multivariate jumps to model systemic risk and its impact on portfolio choice. Bollerslev et al. (2008) use multi-jumps (common

¹FOMC press release, December, 11th 2007, available at [HTTP://WWW.FEDERALRESERVE.GOV/NEWSEVENTS/PRESS/MONETARY/20071211A.HTM](http://www.federalreserve.gov/newsevents/press/monetary/20071211a.htm)

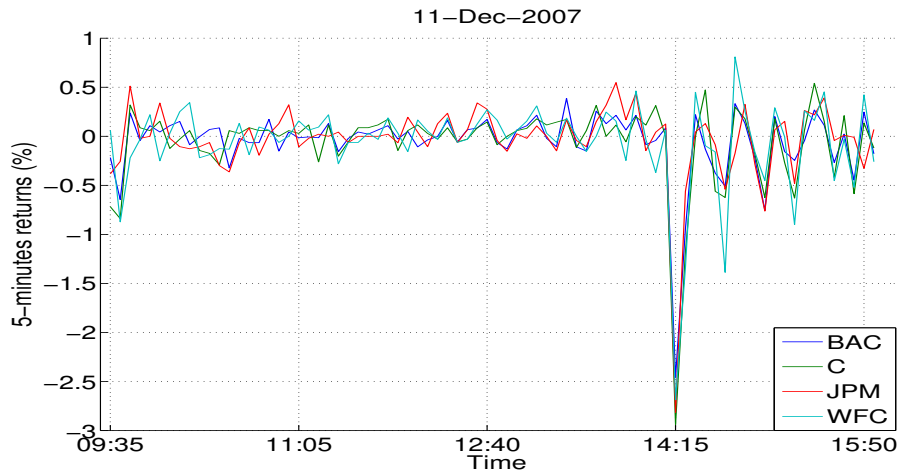


Figure 1: Intraday price changes (log-returns over 5 minutes) of Bank of America (BAC), Citigroup (C), JP Morgan (JPM) and Wells Fargo (WFC) on 11 December 2007. The four banking stocks collapse altogether around 14.15, while a FOMC meeting was taking place. We label this event a *multi-jump*. After the collapse, both volatility and correlation among stocks increases.

jumps, in their terminology) to explain jumps in the aggregated market index and discuss that, for asset allocation, it is more important to be able to detect jumps occurring simultaneously among a large number of assets, since the effect of co-jumps in a pair of assets is negligible in a huge portfolio; Gilder et al. (2014) also study the relation between common jumps and jumps in the market portfolio, and relate common jumps and news. If rare, dramatic multi-jumps can be interpreted as *systemic* events carrying market-wide information on economic fundamentals, their occurrence is also likely to affect the aggregate attitude to risk and thus have an impact on risk premia. For example, Bollerslev and Todorov (2011) empirically supported the view that risk compensation due to large jumps is quite large and time-varying, while Drechsler and Yaron (2011) and Drechsler (2013) highlight the importance of transient non-Gaussian shocks to fundamentals in explaining the magnitude of risk premia. In this paper, we complement this evidence by showing that multi-jumps can be associated with large increases in the variance risk premium.

Despite the statistical, economic and financial importance of multi-jumps, the financial econometrics literature is still missing a formal test to be used as an effective tool for

their detection. A vast literature² concentrated on univariate jump tests. Progress on developing tests for common jumps in a pair of asset prices was started by Barndorff-Nielsen and Shephard (2003). They propose a way to separate out the continuous and co-jump parts of quadratic covariation of a pair of asset prices. Mancini and Gobbi (2012) developed an alternative threshold-based estimator of continuous covariation. Jacod and Todorov (2009) proposed two tests for co-jumps, their approach relying on functionals which depend, asymptotically, on co-jumps only. Finally, Bibinger and Winkelmann (2013) develop a co-jumps test using spectral methods. However, these methodologies apply to the case $N = 2$ only and their generalization to the case $N > 2$ is non-trivial. Bollerslev et al. (2008) propose a test for common jumps in a large panel ($N \rightarrow \infty$) which is based on the pairwise cross-product of intraday returns. In empirical work, detection of multivariate jumps is typically achieved with a simple co-exceedance rule (see, e.g., Gilder et al., 2014), according to which the multi-jump test is the intersection of univariate tests.

We fill this gap in the literature by introducing a novel testing procedure for multi-jumps which naturally applies to the case $N \geq 2$, with N finite. The proposed approach builds on the comparison of two types of suitably introduced smoothed power variations. High values of the test-statistics (which is asymptotically $\chi^2(N)$ under the null) signal the presence of a multi-jump among at least M stocks, with $M \leq N$. The smoothing procedure depends on a bandwidth which can be used to approximately select the desired M , with higher bandwidth values corresponding to higher M . We propose an automated bandwidth selection procedure which can be tuned to get the desired M .

Using simulations of realistic price processes which accommodate for the most relevant empirical features and which are implemented at the 5 – *minutes* frequency (thus making the testing procedure virtually immune from distortions due to the presence of microstructure noise), we show that the proposed procedure i) has desirable size properties; ii) is

²Barndorff-Nielsen and Shephard (2006); Lee and Mykland (2008); Jiang and Oomen (2008); Aït-Sahalia and Jacod (2009) and Christensen et al. (2014), among others.

more powerful and better sized than the Jacod and Todorov (2009) test, which needs a much higher frequency (that is, many more data) to become effective; iii) is remarkably powerful in detecting multi-jumps and iv) strongly outperforms the co-exceedance rule in terms of power.

Results on real data are also encouraging. When applied to 16 liquid US stocks in the period 2003-2012, the test reveals the significant presence of multi-jumps. Not surprisingly, the multi-jumps occurrence rate becomes smaller with larger bandwidth, that is when we increase the minimal order M of stocks jumping jointly. However, multi-jumps with large M (high bandwidth) are rare but important events, which can be always associated with relevant market-wide economic news. This allows to interpret them as *systemic* events affecting the market on a whole.

Importantly, detection of multi-jumps in the stocks reveals additional information with respect to that conveyed by univariate jumps in the index. Indeed, while *theoretically* a multi-jump in the constituents should always correspond to a jump in the index, *empirically* this is not necessarily true since the multi-jumps could have different directions (even if empirical evidence reported in Section 5 documents that this is a quite unlikely event: multi-jumps have typically the same direction) or they could occur in a small subset of stocks, such that the jump in the index could be rather small and hard to detect. These considerations are confirmed by the data: roughly a half of detected multi-jumps in our sample cannot be associated with jumps in the index, unveiling information that univariate jumps could not reveal.

The additional information conveyed by multi-jumps is economically significant. We show that multi-jumps are strongly correlated with large increases in the variance risk premium, while univariate jumps on the index are not. This result is in line with recent theoretical literature, mentioned above, underscoring the impact of jumps in fundamentals on changes in aggregate risk aversion, and the empirical result in Todorov (2010), who makes use of a parametric model to show that price jumps are linked to the vari-

ation in the variance risk-premium. When multi-jumps are used, the association with changes in the variance risk-premium becomes clear-cut also in our fully non-parametric setting. This further indicates that multi-jumps are particularly suitable to test for systemic events, while questioning the usage of index jumps via univariate statistics to this purpose.

To further verify the potential empirical impact of multi-jumps, we show that they have substantial predictive power for volatility and correlations. Both stock correlations and volatilities are found to significantly increase after the occurrence of a multi-jump, thus confirming, on a formal statistical ground, the anecdotal evidence in Figure 1. In particular, the impact of multi-jumps on the correlation coefficient between a given pair of stocks is quite strong, especially when compared to the impact of idiosyncratic co-jumps between the same pair. These results have compelling implications for asset allocation. A risk-averse investor who allocates her wealth in a portfolio of stocks and a risk-free asset is harmed by the presence of multi-jumps in two ways. The first, which could be dealt with the model developed by Das and Uppal (2004), is the change in the optimal allocation strategy due to the presence of multi-jumps with respect to the case without multi-jumps. The second, which we quantify here, is the impact of multi-jump on the covariance matrix of the stocks, which implies an additional utility loss due to the increase in the portfolio variance and the worsening of the diversification potential. The latter effects would induce a less risky, that is less invested in stocks, optimal allocation strategy than that recommended by traditional models.

The remainder of the paper is organized as follows. Section 2 describes the continuous-time jump-diffusion model adopted in the paper. Section 3 explains the formal testing procedure and provides asymptotic results. Section 4 presents results on simulated price dynamics. Section 5 applies the test to real data and contains the empirical results and their implications for asset allocation. Section 6 concludes.

2 Model

Denote the log-prices of an N -dimensional vector of assets by $X = (X^{(i)})_{i=1,\dots,N}$. We assume that stock prices evolve continuously on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \in [0,T]}, \mathcal{P})$ satisfying the usual conditions, and we assume the following dynamics for X , accommodating for continuous (through Brownian motion) and discontinuous (through jumps) shocks.

Assumption 1. X is an N -dimensional Ito semimartingale following:

$$dX_t = a_t dt + \Sigma_t dW_t + dJ_t$$

where a_t (in \mathbb{R}^N) and Σ_t (in $\mathbb{R}^{N \times M}$) are càdlàg adapted processes, W_t is standard multivariate Brownian motion in \mathbb{R}^M and J_t is a finite activity jump process of the form $J_t^{(i)} = \sum_{k=1}^{N_t^{(i)}} \gamma_{\tau_k}^{(i)}$, $i = 1, \dots, N$, and $N_t^{(i)}$ is a non-explosive counting process. Moreover, we assume that the jump sizes are such that, $\forall k = 1, \dots$, we have $\mathcal{P} \left(\gamma_{\tau_k}^{(i)} = 0 \right) = 0$, $i = 1, \dots, N$.

The model, which is very general and encompasses virtually all parametric models typically used in financial applications, allows each component of X to include idiosyncratic jumps (that occur only for a single stock) as well as common jumps among stocks. Define the process

$$\Delta X_t = X_t - X_{t-}, \tag{1}$$

and, as an example, consider the case $N = 3$. The common jumps between $X^{(1)}$ and $X^{(2)}$ satisfy

$$\Delta X_t^{(1)} \Delta X_t^{(2)} = \gamma_t^{1(2)} \gamma_t^{2(1)} \Delta N_t^{12} + \gamma_t^{1(23)} \gamma_t^{2(13)} \Delta N_t^{123},$$

where N^{12} and N^{123} are independent counting processes, while common jumps among all

the three processes³ satisfy

$$\Delta X_t^{(1)} \Delta X_t^{(2)} \Delta X_t^{(3)} = \gamma_t^{1(23)} \gamma_t^{2(13)} \gamma_t^{3(21)} \Delta N_t^{123}.$$

The inference procedure is designed to test the null

$$\sum_{0 \leq t \leq T} \Delta X_t^{(1)} \Delta X_t^{(2)} \Delta X_t^{(3)} = 0$$

against the alternative

$$\sum_{0 \leq t \leq T} \Delta X_t^{(1)} \Delta X_t^{(2)} \Delta X_t^{(3)} \neq 0.$$

Note that the presence of a multi-jump among three assets implies the presence of co-jumps between each pair of them. However, the presence of co-jumps between each pair of assets does not necessarily imply the presence of a multi-jump among them.

We do not explicitly include in the model market microstructure contaminations, since the proposed method is thought to be applied at moderately low frequencies (e.g., five minutes) where the impact of microstructure noise should be negligible. The theory could however be easily extended to include market microstructure noise by adapting our return smoothing technique to preaveraged estimators robust to both jumps and market microstructure noise, as in Podolskij and Vetter (2009) and Hautsch and Podolskij (2013).

The theory could also be extended for infinite activity jumps (see, e.g., Ait-Sahalia and Jacod, 2012 and the references therein), since the test procedure developed below is based on smoothed estimators of integrated variances which have been shown to be consistent even in the presence of this kind of shock, see Mancini (2009) and Mancini and Gobbi (2012).

³To underscore the methodological contribution of this paper, we use the word *co-jump* when the common jump is between two assets, and *multi-jumps* when the common jump is among three or more assets.

3 Multi-jumps inference

Assume to record X in the interval $[0, T]$, with T fixed, in the form of $n+1$ equally spaced observations⁴ and denote by $\Delta = T/n$. Define the evenly sampled logarithmic returns as

$$\Delta_j X = X_{j\Delta} - X_{(j-1)\Delta}, \quad j = 1, \dots, n. \quad (2)$$

In order to formulate the statistical properties of the test, define the following sets:

$$\Omega_T^{MJ,N} = \{\omega \in \Omega \mid \text{the process } \prod_{j=1}^N (\Delta X^{(j)})_t \text{ is not identically } 0\}$$

$$\bar{\Omega}_T^N = \Omega \setminus \Omega_T^{MJ,N}.$$

The set $\Omega_T^{MJ,N}$ contains trajectories with common multi-jumps among *all* N assets in $[0, T]$. The complementary set $\bar{\Omega}_T^N$ contains trajectories without multi-jumps in N stocks; it could however contain jumps and multi-jumps up to $N - 1$ stocks. Testing for multi-jumps is equivalent to testing the following:

$$\mathcal{H}_0 : \left((X_t(\omega))_{t \in [0, T]} \in \bar{\Omega}_T^N \right) \text{ vs. } \mathcal{H}_1 : \left((X_t(\omega))_{t \in [0, T]} \in \Omega_T^{MJ,N} \right). \quad (3)$$

Inference is based on the definition of two newly defined integrated variance estimators which constitute a generalization, particularly suitable to our application, of the truncated realized variance estimator of Mancini (2009). To this purpose we need a definition of a kernel and a bandwidth.

Assumption 2. *A kernel is a function $K(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$, which is differentiable with bounded first derivative almost everywhere in \mathbb{R} , and such that $K(0) = 1$, $0 \leq K(\cdot) \leq 1$ and $\lim_{x \rightarrow \infty} K(|x|) = 0$. The bandwidth process is a sequence $H_{t,n}$ of processes in \mathbb{R}^N*

⁴This requirement can be easily generalized to non-equally spaced observations, if we set $\bar{\Delta} = \max_{i=1, \dots, n} (t_i - t_{i-1})$, where t_i are observation times, and require $\bar{\Delta} \rightarrow 0$, see Remark i) of Theorem 4 in Mancini (2009).

which can be written as $H_{t,n} = h_n \xi_{t,n}$, where h_n is a sequence such that

$$\lim_{n \rightarrow \infty} h_n = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{h_n} \sqrt{\frac{\log n}{n}} = 0, \quad (4)$$

and $\xi_{t,n}$ is a vector of N positive adapted stochastic process on $[0, T]$ which are all a.s. bounded with a strictly positive lower bound.

The bandwidth is written in the form $h_n \xi_{t,n}$ to allow for data-dependent and time-varying bandwidth. Indeed, in our application $\xi_{t,n}$ is the local variance estimated by the observations themselves, see Eq. (31). We call h_n the bandwidth parameter, and provide an automated criterion for its selection in Section B.1 in the Appendix.

We now define two novel jump-robust integrated variance estimators, which are both called *Smoothed Realized Variance*. The first one takes the form

$$\text{SRV}(X^{(i)}) := \sum_{j=1}^n |\Delta_j X^{(i)}|^2 \cdot K \left(\frac{\Delta_j X^{(i)}}{H_{j\Delta,n}^{(i)}} \right), \quad (5)$$

where $X^{(i)}, H^{(i)}$ are the i -th components of the vectors X, H and $K(\cdot)$ and $H_{t,n}$ are the kernel and bandwidth defined in Assumption 2. This estimator coincides with the estimator in Mancini (2009) when $K(x) = I_{\{|x| \leq H_{t,n}\}}$, but allows for a different choice of the kernel. The intuition is however similar to that of Mancini (2009): "smoothed" squared returns $|\Delta_j X^{(i)}|^2 \cdot K \left(\Delta_j X^{(i)} / H_{j\Delta,n}^{(i)} \right)$ are close to squared returns $|\Delta_j X^{(i)}|^2$ when they are small; smoothed squared returns are instead small when returns are large, where the extent of "largeness" is gauged by the bandwidth $H_{j\Delta,n}$. Asymptotically, this procedure annihilates the jumps. The estimator of Mancini (2009) is the most draconian in this respect, since using the indicator function implies that smoothed returns are zero when returns are larger than $H_{j\Delta,n}$ (dubbed *threshold* in Mancini's terminology). The advantage of replacing the indicator function with a smooth kernel is that it provides an estimator which depends smoothly on the bandwidth: This stabilizes the procedure in small samples (by making it less prone to type I and II errors due to erroneous bandwidth

selection) and also eases bandwidth selection.

The following theorem (proof in Appendix A) shows that $\text{SRV}(X^{(i)})$ in Eq. (5) is a jump-robust consistent estimator of integrated variance.

Theorem 3.1. *Let the process X satisfy Assumption 1, and the kernel and bandwidth satisfy Assumption 2. Then, as $n \rightarrow \infty$ we have*

$$\text{SRV}(X^{(i)}) \xrightarrow{p} \int_0^T (\sigma^{(i)})_u^2 du, \quad (6)$$

where $\sigma^{(i)}$ is the volatility of $X_t^{(i)}$.

The following remark introduce a correction to improve the estimator performance in small samples.

Remark 1. (Small Sample Correction) *In order to improve the finite samples unbiasedness of the estimator defined in Eq. (5), it is advisable to normalize it as follows:*

$$\frac{\sum_{j=1}^n |\Delta_j X^{(i)}|^2 \cdot K\left(\frac{\Delta_j X^{(i)}}{H_{j\Delta,n}^{(i)}}\right)}{\Delta \sum_{j=1}^n K\left(\frac{\Delta_j X^{(i)}}{H_{j\Delta,n}^{(i)}}\right)} \xrightarrow{p} \int_0^T (\sigma^{(i)})_u^2 du,$$

since $\Delta \sum_{j=1}^n K\left(\frac{\Delta_j X^{(i)}}{H_{j\Delta,n}^{(i)}}\right) \xrightarrow{p} 1$.

The second estimator takes the form:

$$\widetilde{\text{SRV}}^N(X^{(i)}) := \sum_{j=1}^n |\Delta_j X^{(i)}|^2 \cdot \left(K\left(\frac{\Delta_j X^{(i)}}{H_{j\Delta,n}^{(i)}}\right) + \prod_{k=1}^N \left(1 - K\left(\frac{\Delta_j X^{(k)}}{H_{j\Delta,n}^{(k)}}\right) \right) \right). \quad (7)$$

Returns in Eq. (7) are smoothed as in Eq. (5), but they are also kept similar to the original returns if *all* multivariate returns are big. Thus, even if, when $n \rightarrow \infty$, both smoothing procedures are meant to annihilate jumps, the smoothing in Eq. (7) will let multi-jump survive. This intuition is formalized in the following theorem (proof in

Appendix A), which represents the base for inference and testing.

Theorem 3.2. *Let the process X satisfy Assumption 1, and the kernel and bandwidth satisfy Assumption 2. Then, as $n \rightarrow \infty$,*

$$\widetilde{\text{SRV}}^N(X^{(i)}) \xrightarrow{p} \begin{cases} \int_0^T (\sigma^{(i)})_u^2 du + \sum_{\Delta X_t^{(1)} \dots \Delta X_t^{(N)} \neq 0} (\Delta X_t^{(i)})^2 & \text{on } \Omega_T^{MJ,N} \\ \int_0^T (\sigma^{(i)})_u^2 du, & \text{on } \bar{\Omega}_T^N \end{cases}; \quad (8)$$

where $\sigma^{(i)}$ is the volatility of $X_t^{(i)}$.

Theorems 3.1 and 3.2 introduce a natural estimator for the multi-jumps on each series. By the light of Remark 1 the jump size of stock i corresponding to a multi-jump among all stocks is naturally derived in the following remark.

Remark 2. (Multi-jump Size Estimation)

$$\frac{\widetilde{\text{SRV}}^N(X^{(i)}) - \text{SRV}(X^{(i)})}{\Delta \sum_{j=1}^n K\left(\frac{\Delta_j X^{(i)}}{H_{j\Delta,n}^{(i)}}\right)} \xrightarrow{p} \begin{cases} \sum_{\Delta X_t^{(1)} \dots \Delta X_t^{(N)} \neq 0} (\Delta X_t^{(i)})^2 & \text{on } \Omega_T^{MJ,N} \\ 0, & \text{on } \bar{\Omega}_T^N \end{cases}. \quad (9)$$

In order to define the test statistics, we follow Podolskij and Ziggel (2010) and define a iid $N \times n$ matrix of draws $(\eta_j^i)_{1 \leq i \leq N, 1 \leq j \leq n}$, defined on the canonical extension $(\Omega', \mathcal{F}', (\mathcal{F}')_{t \in [0, T]}, \mathcal{P}')$ of the original probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \in [0, T]}, \mathcal{P})$ and independent from \mathcal{F} . We assume that $\mathbf{E}[\eta_j^i] = 1$ and $\mathbf{Var}[\eta_j^i] = V_\eta < \infty$. Define:

$$\widetilde{\text{SV}}(X^{(i)}) := \sum_{j=1}^n |\Delta_j X^{(i)}|^2 \cdot K\left(\frac{\Delta_j X^{(i)}}{H_{j\Delta,n}^{(i)}}\right) \cdot \eta_j^i, \quad i = 1, \dots, N, \quad (10)$$

and

$$\text{SQ}(X^{(i)}) := \sum_{j=1}^n |\Delta_j X^{(i)}|^4 \cdot K^2\left(\frac{\Delta_j X^{(i)}}{H_{j\Delta,n}^{(i)}}\right), \quad i = 1, \dots, N. \quad (11)$$

The test statistics is then defined as

$$S_{n,N} := \frac{1}{V_\eta} \sum_{i=1}^N \frac{\left(\widetilde{SV}(X^{(i)}) - \widetilde{SRV}^N(X^{(i)}) \right)^2}{SQ(X^{(i)})} \quad (12)$$

and its asymptotics is described in the following Theorem (proof in Appendix A).

Theorem 3.3. *Under Assumption 1 and 2, if $(\eta_j^i)_{1 \leq i \leq N, 1 \leq j \leq n}$ are pairwise independent, as $n \rightarrow \infty$, it holds:*

$$\begin{cases} S_{n,N} \xrightarrow{d} \chi^2(N), & \text{on } \overline{\Omega}_T^N \\ S_{n,N} \xrightarrow{p} +\infty & \text{on } \Omega_T^{MJ,N} \end{cases}; \quad (13)$$

where $\chi^2(N)$ denotes the χ -square distribution with N degrees of freedom.

Theorem 3.3 implies that the statistic $S_{n,N}$ can be used for testing for the presence of multi-jumps. Under \mathcal{H}_0 , the value of $S_{n,N}$ will be distributed as a χ^2 with N degrees of freedom. Under \mathcal{H}_1 , that is in the presence of multi-jumps, it will diverge as the number of observations n increases.

Notice that the test defined in Eq. (12) is of computational order N , in the sense that the computational burden increases linearly with N . In particular, the test does not require the estimation of the covariance between stocks (which would increase the computational burden as N^2).

Following the suggestions of Podolskij and Ziggel (2010) for the univariate jump test, the random variables η_j^i are allowed to take the values $\{1 + \tau, 1 - \tau\}$ with equal probability, so that $V_\eta = \tau^2$. In both Monte Carlo and empirical exercises we use $\tau = 0.05$.

4 Simulation study

In order to simulate the dynamics of realistic prices, we simulate a multivariate model that accommodates correlated prices, stochastic volatility, leverage effect, intraday effects,

idiosyncratic jumps and multi-jumps. All these components have been shown to be present in the dynamics of high-frequency financial prices, and we use realistic parameter values. We do not include market microstructure noise (see the discussion above). The dynamics of the continuous parts of each component are given by the same stochastic differential equations driven by correlated Brownian motions:

$$\begin{aligned} dX_t^{(i)} &= \mu dt + \gamma_t \sigma_t^{(i)} dW_t^{(i)} + dJ_t^{(i)} \\ d\log(\sigma_t^{(i)})^2 &= (\alpha - \beta \log(\sigma_t^{(i)})^2) dt + \eta d\widetilde{W}_t^{(i)}, \end{aligned} \tag{14}$$

where $i = 1, \dots, 16$, $W^{(i)}$ and $\widetilde{W}^{(i)}$ are standard Brownian motions with $\text{corr}(dW^{(i)}, d\widetilde{W}^{(i)}) = \widetilde{\rho}$; $\sigma_t^{(i)}$ are stochastic volatility factors and γ_t represent intraday effects. The Brownian motions $W^{(i)}$ driving the price dynamics can be correlated, as specified below. The pure jump parts of $X^{(i)}$ are different compound Poisson processes.

The parameters of the model are taken to be as estimated by Andersen et al. (2002) on S&P500 prices: $\mu = 0.0304$, $\alpha = -0.012$, $\beta = 0.0145$, $\eta = 0.1153$, $\widetilde{\rho} = -0.6127$; where the parameters are expressed in daily units and returns are in percentage. The intraday effects are given by:

$$\gamma_t = \frac{1}{0.1033}(0.1271 \cdot t^2 - 0.1260 \cdot t + 0.1239),$$

as estimated on S&P500 intraday returns. In our simulations, we always have $t \in [0, 1]$, with initial values for prices and volatility taken from the last simulated day.

The model (14) is discretized with the Euler scheme, using discretization step of $\Delta = \frac{1}{80}$ which roughly corresponds to 5-minutes returns for a trading day of 6.5 hours ($n = 80$). We generate samples of 1,000 days with different specifications for the jump processes $dJ_t^{(i)}$.

